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Inverse Transient Quasi-Static Thermal Stresses in a Thin Rectangular Plate

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Abstract

In this paper the solution of inverse transient quasi-static thermal stresses in a thin rectangular plate occupying the space $-a \leq x \leq a$, $-b \leq y \leq b$ has been determined. The expression of the temperature distribution, unknown temperature, stresses function, displacement component and thermal stresses component are obtained in terms of Bessel's function, and the results are illustrated numerically and graphically.

Key words: Thin rectangular plate, inverse transient quasi-static thermal stress, Fourier transform, Laplace transform, Inverse Fourier and Laplace transform

1 Introduction

A consistent effort in the past has been made by numerous workers to study various aspects of thermal stresses in different bodies. Tanigawa Y., Ishihara M., Morishita H., and Kawamura R. [7] have studied theoretical analysis of two dimensional thermoelastoplastic bending deformation of plate subjected to partially distributed heat supply. Yoshinobu Tanigawa and Yasuo Komatsubara [9] have discussed thermal stress analysis of a rectangular plate and its thermal stress intensity factor for compressive stress field. Ishihara M., Tanigawa Y., Kawamura R. and Noda N. [3] have studied theoretical analysis of residual stress by removing the heat supply. Further, Vihak V.M., Yuzvyak M.Y. and Yasinkij A.V. [8] have investigated the solution of the plane thermo-elasticity problem for a rectangular domain. Adams R.J. and Bert C.W. [1] have determined thermoelastic vibration of a laminated rectangular plate subjected to a thermal shock.

N. L. Khobragade and Deshmukh K.C. have studied an inverse quasi-static thermal deflection problem for a thin clamped circular plate [11].

In this article, an attempt is being made to find the solution of the inverse transient problem of Quasi-static thermal stresses in a rectangular plate. In order to obtain the solution of the governing equation, which is a partial differential equation, the following procedures of analysis have been adopted.

1. Normalizing of the governing partial differential equation subject to appropriate initial, boundary and interior conditions.
2. Taking the finite Fourier and Laplace transform of the resulting equation with respect to time.
3. Achieving the inverse Fourier and inverse Laplace transform by means of complex contour integration residue and convolution theorem.
4. The solution and expressions of the temperature, unknown temperature, stress function, displacement and stress components are obtained in terms of Bessel's function, and the results are illustrated numerically and graphically.

The results obtained may be useful in solving engineering problems, particularly for industrial machines subjected to heating such as the main shaft of a lathe, turbines and the rolls of the rolling mill.

2 Statement of inverse heat conduction problem

Consider a thin rectangle plate occupying the space D : $-a \le x \le a$, $-b \le y \le b$. Initially the temperature of plate is kept at zero. The differential equation governing transient temperature distribution T(x,y,t) in rectangular plate is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{K}.\frac{\partial T}{\partial t} \qquad -a \le x \le a,\ -b \le y \le b,\ t > 0 \qquad (1)$$

subject to the initial boundary and interior conditions as

$$T(x, y, t)_{t=0} = 0 \qquad (2)$$

$$\left(\frac{\partial T}{\partial x}\right)_{x=\pm a} = 0 \tag{3}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=-b} = 0 \tag{4}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=b} = g(x,t) \quad \text{unknown} \tag{5}$$

$$T(x,y,t)_{y=\xi} = -f(x) \quad \text{known } -b \le \xi \le b \tag{6}$$

where T, K are the temperature change and thermal diffusivity. The Airy's stress function U satisfies the equation as in [9]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 U = -\alpha E\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T \tag{7}$$

where α and E are the linear coefficients of the thermal expansion and Young's modulus of elasticity of the materials of the rectangular plate respectively.

The displacement components U_x and U_y in the x and y direction represented in the integral form and the stress components in terms of U are given by

$$U_x = \int\left(\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial x^2}\right) + \alpha T\right).dx \tag{8}$$

$$U_y = \int\left(\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right) + \alpha T\right).dy \tag{9}$$

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \tag{10}$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \tag{11}$$

and $$\sigma_{xy} = \frac{\partial^2 U}{\partial x \partial y} \tag{12}$$

where ν is the Poisson's ratio of the material of the plate.

The equation (1) to (12) constitutes the mathematical formulations of the problem under consideration.

3 Solution of the inverse heat conduction problem

Determination of temperature T(x,y,t) and unknown temperature gradient g(x,t)

On applying finite Fourier and Laplace transform to the equation (1) to (6) and then using their inversions, one obtains the expression of the temperature distribution and unknown temperature gradient respectively as,

$$T(x,y,t) = \sum_{n,p=1}^{\infty} \frac{-2\xi_p.\cos\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{13}$$

$$g(x,t)=\sum_{n,p=1}^{\infty}\frac{2\xi_p^2.\sin(\xi_p 2b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{14}$$

where $G(m,n,t)=\left[1-\exp\left(-K\left(\xi_n^2+\xi_p^2\right)t\right)\right]$

ξ_n is the Fourier transform parameter and Kernal $K_0(\xi_n.x)$ defined as,

$$K_0(\xi_n,x)=\frac{R_0(\xi_n,x)}{\sqrt{N_x}} \tag{15}$$

$$R_0(\xi_n,x)=\cos\xi_n(x+a)$$

$$\frac{1}{\sqrt{N_x}}=\frac{2}{a}$$

where ξ_n is the positive root of the transcendental equation.

$$\sin 2\xi_n x=0 \tag{16}$$

4 Determination of stress function and displacement component

Substituting the value of temperature function from (13) in (7), we obtain

$$U(x,y,t)=-2\alpha E\sum_{n,p=1}^{\infty}\frac{\xi_p.\cos\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)^2}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{17}$$

Substituting the value of T and U from (13) in (17) in (8) and (9), we obtain

$$U_x=-2\alpha(1+v)\sum_{n,p=1}^{\infty}\frac{\xi_n\xi_p.\cos\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)^2}.\frac{\sin\xi_n(x+a)}{Nx}.G(m,n,t) \tag{18}$$

$$U_y=-2\alpha(1+v)\sum_{n,p=1}^{\infty}\frac{\xi_p^2.\sin\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)^2}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{19}$$

5 Determination of stress components

Using (17) in (10) (11) and (12), the stress functions are obtained as

$$\sigma_{xx}=2\alpha E\sum_{n,p=1}^{\infty}\frac{\xi_p^3.\cos\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)^2}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{20}$$

$$\sigma_{yy}=2\alpha E\sum_{n,p=1}^{\infty}\frac{\xi_n^2\xi_p.\cos\xi_p(y+b)}{(\xi+b)\sin(\xi_p(\xi+b))}\frac{\overline{f}(\xi_n)}{(\xi_n^2+\xi_p^2)^2}.\frac{K_0(\xi_n,x)}{Nx}.G(m,n,t) \tag{21}$$

$$\sigma_{xy} = -2\alpha E \sum_{n,p=1}^{\infty} \frac{\xi_n \xi_p^2 . \sin \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi + b))} \cdot \frac{\overline{f}(\xi_n)}{(\xi_n^2 + \xi_p^2)^2} \cdot \frac{\sin \xi_n (x+a)}{Nx} . G(m,n,t) \qquad (22)$$

6 Special case and Numerical results

Set $f(x) = ax - \frac{x^2}{2}$ (23)

Applying the finite Fourier transform to the equation (23) one obtains

$$\overline{f}(\xi_n) = \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \qquad (24)$$

Now for finding out the solution of inverse transient quasi-static thermal stress, steel (SN50C) has been taken as the material of the plate having the following properties :

$\xi_1 := 1.309$ $\xi_2 := 3.927$ $\xi_3 := 4.974$ $\xi_4 := 6.021$ $\xi_5 := 8.116$ $\xi_6 := 10.21$ $\xi_7 := 13.352$

$\xi_8 := 15.97$ $\xi_9 := 17.017$ $\xi_{10} := 19.111$ $\alpha := 11.610^{-6} \cdot K^{-1}$ $v := 0.281$ $c := 0.03$

$t := 90$ $x := 3$ $y := 0.1, 0.2 .. 0.6$ $a := 2$ $b := 1$

$\delta = -2\alpha E$ $\beta = -2\alpha(1+\nu)$ $\gamma = 2\alpha E$ $K = 15.9 \times 10^{-6}\ m^2 s^{-1}$
E=215 GPa

Substituting (24) in (13), (14), (17), (18), (19), (20), (21) and (22)

$$T(x,y,t) = \sum_{n,p=1}^{\infty} \frac{-2\xi_p . \cos \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi + b))} \cdot \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \qquad (25)$$

$$g(x,t) = \sum_{n,p=1}^{\infty} \frac{2\xi_p^2 . \sin(\xi_p 2b)}{(\xi + b)\sin(\xi_p(\xi + b))} \cdot \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \qquad (26)$$

$$U(x,y,t)/\delta = \sum_{n,p=1}^{\infty} \frac{\xi_p . \cos \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi + b))} \cdot \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \qquad (27)$$

$$U_x/\beta = \sum_{n,p=1}^{\infty} \frac{\xi_n \xi_p . \cos \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi+b))} . \frac{\sin \xi_n (x+a)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \quad (28)$$

$$U_y/\beta = \sum_{n,p=1}^{\infty} \frac{\xi_p^2 . \sin \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi+b))} . \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \quad (29)$$

$$\sigma_{xx}/\gamma = \sum_{n,p=1}^{\infty} \frac{\xi_p^3 . \cos \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi+b))} . \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \quad (30)$$

$$\sigma_{yy}/\gamma = \sum_{n,p=1}^{\infty} \frac{\xi_n^2 \xi_p . \cos \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi+b))} . \frac{K_0(\xi_n, x)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \quad (31)$$

$$\sigma_{xy}/\gamma = \sum_{n,p=1}^{\infty} \frac{-\xi_n \xi_p^2 . \sin \xi_p (y+b)}{(\xi + b)\sin(\xi_p(\xi+b))} . \frac{\sin \xi_n (x+a)}{Nx(\xi_n^2 + \xi_p^2)^2} . G(m,n,t) \int_{-a}^{a} \left(ax - \frac{x^2}{2} \right) \cos \xi_n (x+a) . dx \quad (32)$$

7 Graphs

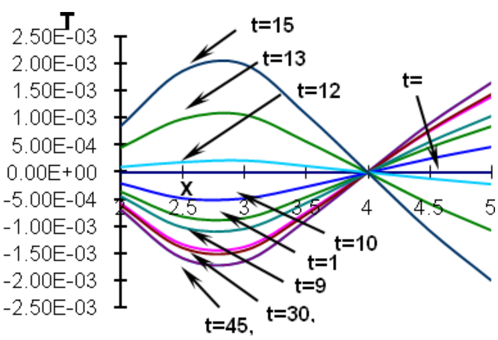

Fig. 1 : Temperature distribution T versus x at t = 15, 30, …, 150 for fixed y = 0.05

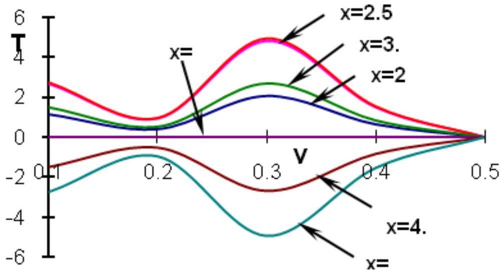

Fig. 2 : Temperature distribution T versus y at x = 2, 2.5, …, 5for fixed t = 15

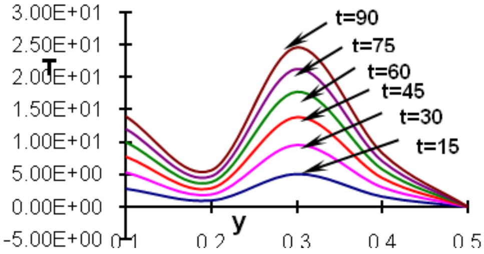

Fig. 3 : Temperature distribution T versus y at t = 15, 30 … 60 for fixed value of x = 3

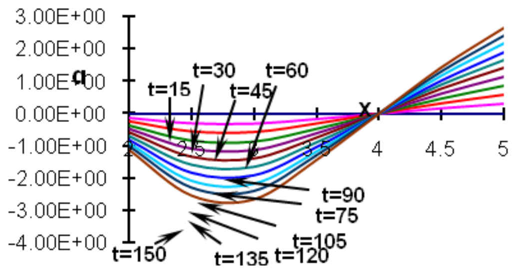

Fig. 4 : Unknown Temperature distribution g versus x at t = 15, 30 … 150 for fixed value of y = 0.5

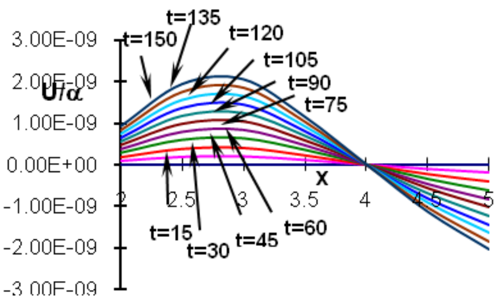

Fig. 5 : Stress function U/α versus x at t = 15, 30, …, 150 for fixed y = 0.5

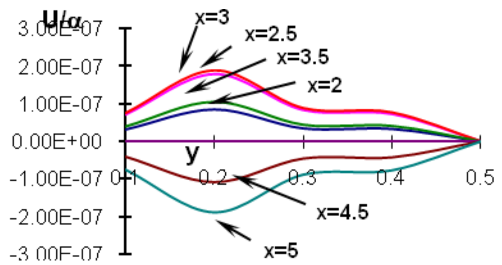

Fig. 6 : Stress function U/α versus y at x = 2, 2.5 ... 5 for fixed t = 15

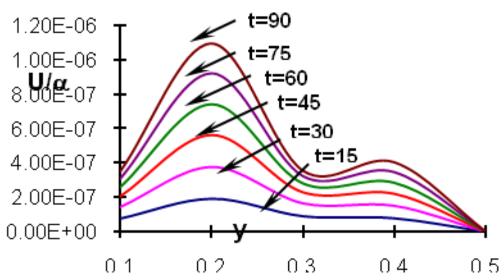

Fig. 7 : Stress function U/α versus y at t = 15, 30 ... 90 for fixed x = 3

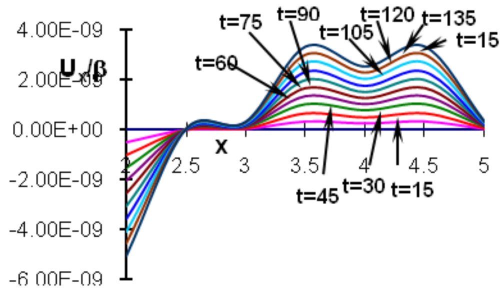

Fig. 8 : Displacement Component U_x/β versus x at t = 15, 30, …, 150for fixed y = 0.5

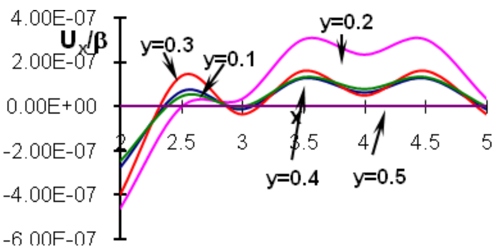

Fig. 9 : Displacement Component U_x/β versus x at y = 0.1, 0.2, …, 0.5for fixed t = 15

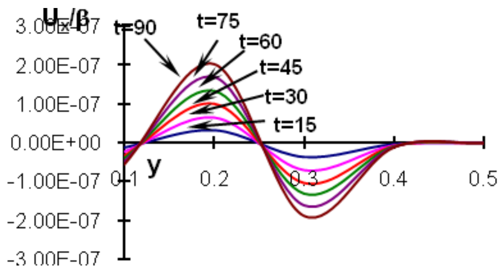

Fig. 10 : Displacement Component U_x/β versus y at t = 15, 30, …, 90for fixed x = 3

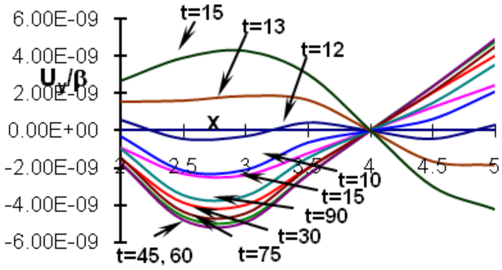

Fig. 11 : Displacement Component U_y/β versus x at t = 15, 30, …, 150for fixed y = 0.5

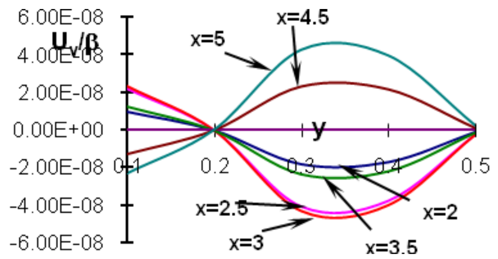

Fig. 12 : Displacement Component Uy/β versus y at x = 2, 2.5, …, 5 for fixed t = 15

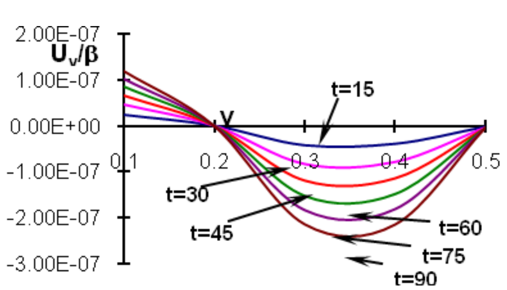

Fig. 13 : Displacement Component U_y/β versus y at t = 15, 30, …, 90 for fixed x = 3

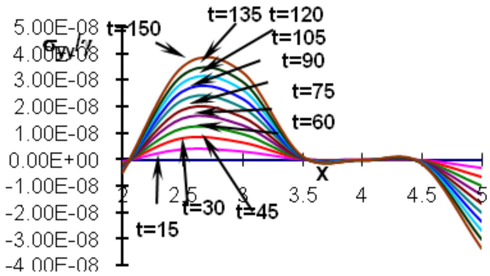

Fig. 14 : Stresses component σyy/γ versus x at t = 15, 30, …, 150 for fixed y = 0.5

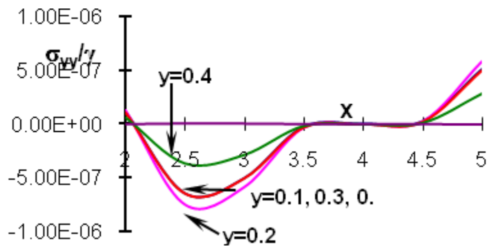

Fig. 15 : Stresses component σ_{yy}/γ versus x at y = 0.1, 0.2, …, 0.5 for fixed t = 15

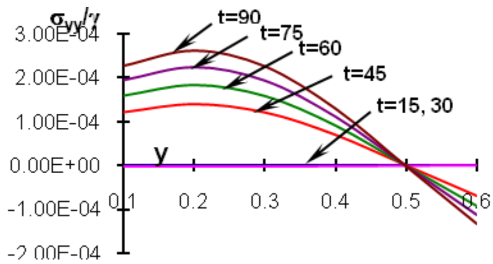

Fig. 16 : Stresses component σ_{yy}/γ versus y at t = 15, 30, …, 90 for fixed x = 3

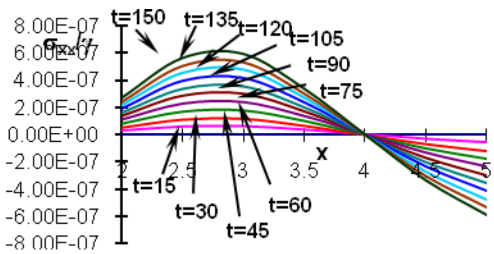

Fig. 17 : Stresses component σ_{xx}/γ versus x at t = 15, 30, …, 150 for fixed y = 0.5

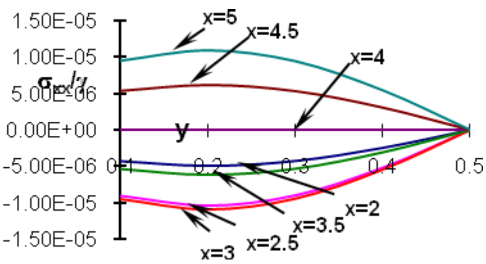

Fig. 18 : Stresses component σ_{xx}/γ versus y at x = 2, 2.5, …, 5 for fixed t = 15

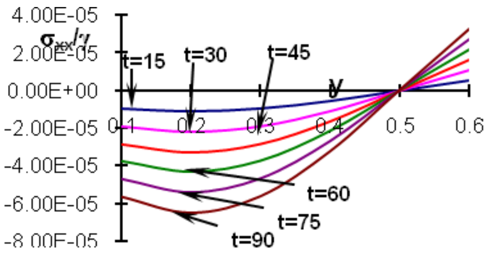

Fig. 19 : Stresses component σ_{xx}/γ versus y at t = 15, 30, …, 90 for fixed x = 3

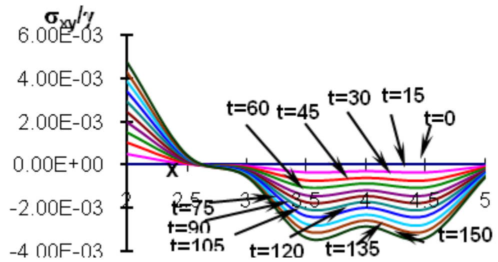

Fig. 20 : Stresses component σ_{xy}/γ versus x at t = 15, 30, …, 150 for fixed y = 0.5

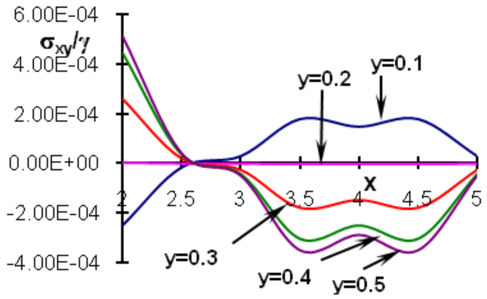

Fig. 21 : Stresses component σ_{xy}/γ versus x at y = 0.1, 0.2, …, 0.5 for fixed t = 15

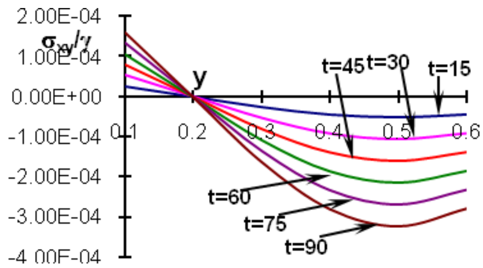

Fig. 22 : Stresses component σ_{xy}/γ versus y at t = 15, 30, …, 90 for fixed x = 3

8 Discussion

In this paper, equations (25) through (32) have been calculated on our observations using MathCAD, graphs have been plotted accordingly as per the results obtained, and conclusions are being drawn.

Initially the temperature of the rectangular plate has been determined by using the conditions given in the problem and applying finite Fourier and Laplace transform and its inverse. Thereafter, the unknown temperature has been found out using the temperature results at y = b. Thus, the value of stress function of the material of the plate is found using temperature T, linear coefficients of the thermal expansion α, Young's modulus of elasticity E, and Poisson's ratio of the material v. Finally, the displacement component has been arrived at using the stress function; and lastly, the stress component in terms of U has been found.

Now, thermal diffusivity and thermal conductivity are two important thermal properties that enter the differential equation of heat conduction. Therefore, accuracy of the value chosen for these properties affects the accuracy of the results in heat-conduction problems. In isotropic rectangular plate the change of temperature does not lead to any change in shear angles, except the stress and strain of the plate. These properties are clearly reflected in the above plotted graphs.

In Fig. 1 through Fig. 22, the temperature distribution T, unknown temperature g, stress function U, displacement component U_x and U_y, and stress components σ_{xx},

σ_{yy}, σ_{xy} of the rectangular plate versus x for different values of y and t, have been plotted. The temperature distribution T, unknown temperature g, stress function U, displacement component U_x and U_y, and stress components σ_{xx}, σ_{yy}, σ_{xy} of the rectangular plate versus y for different values of x and t, has also been considered. In this problem of inverse transient quasi-static thermal stress in thin rectangular plate, the condition that has been given, is that, its base is subjected to heat flux and initially the temperature of the plate has been kept at zero. Here, we have considered steel plate (SN50C) as the metal, and hence the graph shows a particular pattern resembling the properties of steel plate (SN50C).

Thus the following conclusion can be drawn:

1. The temperature distribution increases as the time increases, and is at its optimum at time t = 150 for different values of t versus x, for fixed y = 0.05; and each line coincides at x = 4. But, on increasing the time t above 150, the graph refuses to show any further consistency (refer fig. 1).
2. Same pattern in the graphical representation can be observed in unknown temperature (Fig.4), stress function U (Fig.5), displacement component (Fig.8 & 11), and stress component (Fig.14, 17 & 20).
3. The temperature distribution is optimum at x = 3 (refer Fig.2), after which it starts declining at different value of x versus y for fixed time t = 15. The same pattern is observed in stress function U (Fig.6) and displacement component U_y (Fig.12). All lines coincide at y = 0.5. The stress component (Fig.18) shows a reverse pattern in the graph, due to the negative sign in the stress component.
4. The temperature distribution rapidly increases as time t increases for fixed value of x versus y and coincides at y = 0.5 (refer Fig.3). The same pattern is observed in stress function U (Fig.7), displacement component U_x (Fig.10 & 13), and stress component (Fig.16, 19 & 22).
5. Displacement component is maximum at y = 0.2 for different values of y versus x for fixed time t = 15 (refer Fig.9). Here every line coincides at x = 5. But, a reverse pattern in the graph obtained in stress component σ_{yy} and σ_{xy} (Fig.15 & 21) due to its negative sign.

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