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# Adomian Decomposition Method for Dispersion Phenomena Arising in Longitudinal Dispersion of Miscible Fluid Flow through Porous Media

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**Abstract**

In this paper a theoretical model is developed for the dispersion problem in porous media in which the flow is one dimensional and the average flow is unsteady. The Numerical Solution and graphical illustration of the dispersion problem is presented by means of Adomian Decomposition method for nonlinear partial differential equations and **Mathematica** 7.0.

**Mathematics Subject Classification**: 81U30, 76S05

**Keywords**: Adomian Decomposition method, Miscible fluid, Dispersion, Flows in porous media.

## 1. Introduction

In a miscible displacement process a fluid is displaced in a porous medium by another fluid that is miscible with the first fluid. Miscible displacement in porous media plays a prominent role in many engineering and science fields such as oil recovery in petroleum engineering, contamination of ground water by waste product disposed under ground movement of mineral in the soil and recovery of spent liquors in pulping process. Among Many flow problems in porous media, one involves fluid mixtures called miscible fluids. A miscible fluid is a single phase fluid consisting of several completely dissolved homogenous fluid species, a distinct fluid-fluid interface doesn't exist in a miscible fluid. The flow of miscible fluid is an important topic in petroleum industry; an enhanced recovery technique in oil reservoir involves injecting a fluid (solvent) that will dissolve the reservoir's oil.

These problems of dispersion have been receiving considerable attention from chemical, environmental and petroleum engineers, hydrologists, mathematicians and soil scientists. Most of the works reveal common assumption of homogenous porous media with constant porosity, steady seepage flow velocity and constant dispersion coefficient. For such assumption Ebach and White [4] studied the longitudinal dispersion problem for an input concentration that varies periodically with time. Al-Niami and Rushton [1] studied the analysis of flow against dispersion in porous media. Hunt [6] applied the perturbation method to longitudinal and lateral dispersion in no uniform seepage flow through heterogeneous aquifers. Meher and Mehta [8, 9] studied the Dispersion of Miscible fluid in semi infinite porous media with unsteady velocity distribution from different point of view.

The present paper discusses the approximate solution of the nonlinear differential equation for longitudinal dispersion phenomena which takes places when miscible fluids mix in the direction of flow. The mathematical formulation of the problem yields a non linear partial differential equation. Solution has been obtained by using Adomian decomposition method.

## 2. Mathematical Formulation and Solution of the Problem

The problem is to find the concentration as a function of time'$t$' and position '$x$' as the two miscible fluid flow through porous media on either sides of the mixed region . The single fluid equation describes the motion of fluid [14]. Here the mixing takes place longitudinally as well as transversely at $t = 0$ and a dot of fluid having $[C_0]$ concentration is injected over the phase. The dot moves in the direction of flow as well as perpendicular to the flow. Finally it takes the shape of the ellipse with a different concentration $[C_n]$.

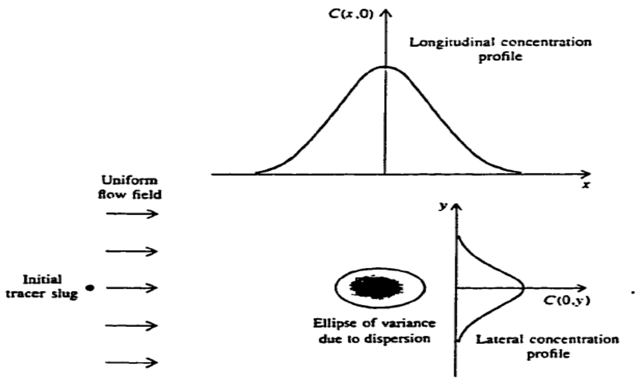


**Fig: 1. (Dispersion of an instantaneous point source in a uniform flow field)**

According to Darcy's law the equation of continuity for the mixture in case of incompressible fluids is given by $\frac{\partial \rho}{\partial t}+\nabla.(\rho \bar{v})=0$ (1)

Where '$\rho$'is the density for mixture and '$\bar{v}$'is the pore seepage velocity.

The equation of diffusion for a fluid flow through a homogeneous porous medium with out increasing or decreasing the dispersing material Polubarinova [10] is given by

$$\frac{\partial C}{\partial t}+\nabla.(C\bar{v})=\nabla.\left[\rho \bar{D}\nabla\left(\frac{C}{\rho}\right)\right] \quad (2)$$

Where 'C' is the concentration of a fluid in a porous media. $D$ is the Coefficient of dispersion with nine components $D_{ij}$. In a laminar flow for an Incompressible fluid through homogeneous porous medium, density '$\rho$' is constant. Then equation (2) becomes,

$$\frac{\partial C}{\partial t}+\bar{v}.\nabla C=\nabla.(\bar{D}\nabla C) \quad (3)$$

Let us assume that the seepage velocity $\bar{v}$ is along the x- axis, then $\bar{v}=u(x,t)$ and the non zero components will be $D_{11}\approx D_L=\gamma$ (coefficient of longitudinal dispersion) and other Components will be zero Polubarinova [10].

Equation (3) becomes

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=\gamma\frac{\partial^2 C}{\partial x^2} \quad (4)$$

Where u is the component velocity along x-axis which is time dependent as well as concentration along x axis in $x\geq 0$ direction and $D_L>0$ and it is the cross sectional flow velocity in porous media. $u=\frac{C(x,t)}{C_0}$, Where $x>0$ and for $C_0\cong 1$ by Mehta[9].Equation (4) becomes

$$\frac{\partial C}{\partial t}+C\frac{\partial C}{\partial x}-\gamma\frac{\partial^2 C}{\partial x^2}=0 \quad (5)$$

This is the non linear Burger's equation for longitudinal dispersion of miscible fluid flow through porous media.

The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants $C_0$ . The porous medium is considered as nonadsorbing. Consider the input concentration is $C_0$ .The governing partial differential equation (5) for longitudinal hydrodynamic dispersion with in a semi-infinite nonadsorbing porous medium in a unidirectional flow field in which $\gamma$ is the longitudinal dispersion coefficient, C is the average cross-sectional concentration, *u* is the unsteady seepage velocity, x is a coordinate parallel to flow and *t* is time.

The initial and boundary conditions are
$$C(x,0) = C_0(x) = e^{-x}, x \geq 0$$
$$C(0,t) = C_1(t) = 1, t \geq 0 \ \ (say) \tag{6}$$

Since Concentration is decreasing as x with distance x. Therefore for the sake of convenience f(x) is considered as negative exponential function Mehta [9].

## 3. Adomian Decomposition Method A Theoretic Approach

We solve equation (4) for $L_t C(x,t)$ and $L_x C(x,t)$ separately and we get

$$L_t C(x,t) = \gamma L_x C(x,t) - NC(x,t) \tag{7}$$

$$L_x C(x,t) = \gamma^{-1}(L_t C(x,t) + NC(x,t) \tag{8}$$

Let $L_t^{-1} and L_x^{-1}$ be the inverse operators of $L_t C(x,t)$ and $L_x C(x,t)$ respectively, given by the form: $L_t^{-1} = \int (.)\, dt$ and $L_x^{-1} = \iint (.) dxdx$ (9)

Then operating both sides of equation (7) and (8) with the inverse operators (9) we obtain

$$C(x,t) = A + L_t^{-1}(\gamma \frac{\partial^2 C}{\partial x^2} - C\frac{\partial C}{\partial x}) \tag{10}$$

$$C(x,t) = B + \gamma^{-1} L_x^{-1}(\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x}) \tag{11}$$

Where A and B can be determined subjected to the corresponding initial and boundary condition (6) and we obtain: A= $e^{-x}$, $B = C_1$=1 (12)

Now adding (10) and (11) and dividing by 2, we get the following form

$$C(x,t) = \frac{1}{2}\left[(A + B) + L_t^{-1}(\gamma \frac{\partial^2 C}{\partial x^2} - C\frac{\partial C}{\partial x}) + \gamma^{-1} L_x^{-1}(\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x})\right]$$
$$= \left(\frac{e^{-x} + 1}{2}\right) + \frac{1}{2}\left[L_t^{-1}(\gamma \frac{\partial^2 C}{\partial x^2} - C\frac{\partial C}{\partial x}) + \gamma^{-1} L_x^{-1}(\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x})\right] \tag{13}$$

We write the parameterized form of (13)

$$C(x,t) = C_0(x,t) + \frac{\lambda}{2}\left[ L_t^{-1}(\gamma \frac{\partial^2 C}{\partial x^2} - C\frac{\partial C}{\partial x}) + \gamma^{-1} L_x^{-1}(\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x}) \right] \quad (14)$$

and the parameterized decomposition forms of C and NC as

$$C(x,t) = \sum_{n=0}^{\infty} \lambda^n C_n(x,t) \quad (15)$$

and $C(x,t)\frac{\partial C}{\partial x}$ =NC(x,t)= $\sum_{n=0}^{\infty} \lambda^n A_n$ (16)

Where $A_n$ the Adomian's special polynomials are to be determined. Here the parameter $\lambda$ looks like a perturbation parameter; but actually is not a perturbation parameter; it is used only for grouping the terms. Now substitution of (15) and (16) into (14) gives

$$\sum_{n=0}^{\infty} \lambda^n C_n(x,t) = C_0(x,t) + \frac{\lambda}{2}\left[ L_t^{-1}(\gamma \frac{\partial^2 \sum_{n=0}^{\infty} \lambda^n C_n}{\partial x^2} - \sum_{n=0}^{\infty} \lambda^n A_n) + \gamma^{-1} L_x^{-1}(\frac{\partial \sum_{n=0}^{\infty} \lambda^n C_n}{\partial t} + \sum_{n=0}^{\infty} \lambda^n A_n) \right] \quad (17)$$

If we compare like power terms of $\lambda$ from both sides of equation (17) and taking under consideration that parameter $\lambda$ is being proved that has the unique value $\lambda$ =1 Cherruault[3] we get

$$C_0(x,t) = \left( \frac{e^{-x} + 1}{2} \right)$$

$$C_1(x,t) = \frac{1}{2}\left[ L_t^{-1}(\gamma \frac{\partial^2 C_0}{\partial x^2} - A_0) + \gamma^{-1} L_x^{-1}(\frac{\partial C_0}{\partial t} + A_0) \right]$$

$$C_2(x,t) = \frac{1}{2}\left[ L_t^{-1}(\gamma \frac{\partial^2 C_1}{\partial x^2} - A_1) + \gamma^{-1} L_x^{-1}(\frac{\partial C_1}{\partial t} + A_1) \right]$$

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$$C_{n+1}(x,t) = \frac{1}{2}\left[ L_t^{-1}(\gamma \frac{\partial^2 C_n}{\partial x^2} - A_1) + \gamma^{-1} L_x^{-1}(\frac{\partial C_n}{\partial t} + A_1) \right], \text{ n=0,1,2,3……n} \quad (18)$$

Next we determine Adomian's special Polynomials $A_n$'s .

## 4. Determination of Adomian's Special Polynomials

The $A_n$ polynomials are determined in such a way that each $A_n$ depend only on $C_0, C_1, .........C_n$ for n = 0, 1, 2, 3….n, i.e.
$A_0 = A(C_0), A_1 = A_1(C_0, C_1), A_2 = A_2(C_0, C_1, C_2)$ etc. In order to do this we substitute (15) in to (16) and we have

$$NC(x,t) = C\frac{\partial C}{\partial x} = (C_0 + \lambda C_1 + \lambda^2 C_2 + .....)(\frac{\partial C_0}{\partial x} + \lambda\frac{\partial C_1}{\partial x} + \lambda^2\frac{\partial C_2}{\partial x} + .....)$$
$$= C_0\frac{\partial C_0}{\partial x} + \lambda(C_0\frac{\partial C_1}{\partial x} + C_1\frac{\partial C_0}{\partial x}) + \lambda^2(C_0\frac{\partial C_2}{\partial x} + C_1\frac{\partial C_1}{\partial x} + C_2\frac{\partial C_0}{\partial x}) + \lambda^3(.......) \quad (19)$$

From (16) we conclude that the Adomian polynomials have the following form:

$$A_0 = C_0\frac{\partial C_0}{\partial x}$$
$$A_1 = C_0\frac{\partial C_1}{\partial x} + C_1\frac{\partial C_0}{\partial x}$$
$$A_2 = C_0\frac{\partial C_2}{\partial x} + C_1\frac{\partial C_1}{\partial x} + C_2\frac{\partial C_0}{\partial x} \quad (20)$$

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Hence, the polynomial $A_0$ has the following form:

$$A_0 = -\left(\frac{e^{-2x} + e^{-x}}{4}\right) \Rightarrow C_1 = \frac{1}{2}[-\frac{\gamma^{-1}}{16}e^{-2x} - \frac{\gamma^{-1}}{4}e^{-x} + \frac{\gamma}{2}e^{-x}t + \left(\frac{e^{-2x} + e^{-x}}{4}\right)\text{t}]$$

If we suggest as a solution of C as an approximation of only two terms of the form

$$C = C_0 + C_1 = \left(\frac{e^{-x} + 1}{2}\right) + \frac{1}{2}[-\frac{\gamma^{-1}}{16}e^{-2x} - \frac{\gamma^{-1}}{4}e^{-x} + \frac{\gamma}{2}e^{-x}t + \left(\frac{e^{-2x} + e^{-x}}{4}\right)\text{t}] \quad (21)$$

We use **Mathematica 7.0** in order to get numerical results…..

```
C0=1/2{Exp[-x]+1}
xC0=D[C0,x]
tC0=D[C0,t]
dC0=D[C0,x,x]
a=Integrate[-{Exp[-2x]+Exp[-x]}/4,x,x]
b=Integrate[n/2{Exp[-x]}+{Exp[-2x]+Exp[-x]}/4,t]
u1=Expand[1/2(b+n^-1a)]
u=Expand [C0+C1]
p=Expand[C/.{t→.01,n→1}]
Plot[{p},{x,0,.9}]
```

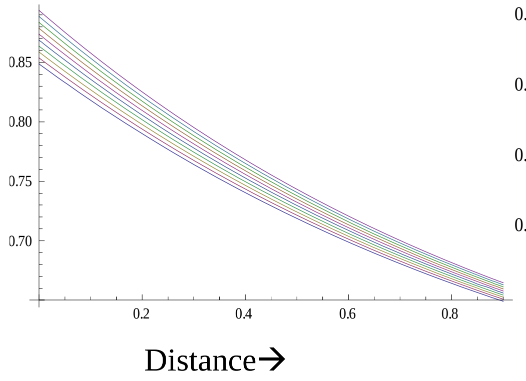
**TABLE-1**

| x/t | 0.001 | 0.002 | 0.003 | .004 | .005 | .006 | .007 | .008 | .009 | .01 |
|---|---|---|---|---|---|---|---|---|---|---|
| .1 | .81417 | .814612 | .815054 | .815495 | .81594 | .816378 | .81682 | .81726 | .817704 | .81815 |
| .2 | .78647 | .786858 | .787249 | .78764 | .78803 | .788421 | .78881 | .78920 | .789594 | .78999 |
| .3 | .76100 | .761349 | .761696 | .762042 | .76239 | .762735 | .76308 | .76343 | .763774 | .76412 |
| .4 | .73764 | .737944 | .738251 | .738559 | .73887 | .739174 | .73948 | .73979 | .740096 | .74040 |
| .5 | .71623 | .7165 | .716773 | .717047 | .71732 | .717593 | .71786 | .71814 | .718414 | .71869 |
| .6 | .69664 | .696879 | .697122 | .697366 | .69761 | .697853 | .69809 | .69834 | .698583 | .69883 |
| .7 | .67873 | .678947 | .679164 | .679382 | .67960 | .679816 | .68003 | .68025 | .680467 | .68068 |
| .8 | .66238 | .662577 | .66277 | .662964 | .66316 | .663352 | .66354 | .66374 | .663933 | .66413 |
| .9 | .64747 | .647644 | .647817 | .647991 | .6482 | .648337 | .64851 | .64868 | .648856 | .64903 |
| 1 | .63388 | .634035 | .63419 | .634345 | .6345 | .634655 | .63481 | .634965 | .635119 | .63527 |

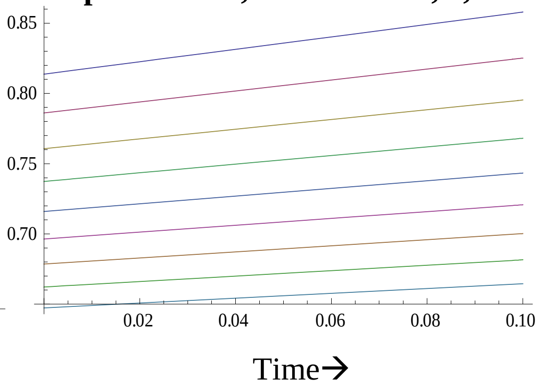
**TABLE-2**

| x/t | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 | .1 |
|---|---|---|---|---|---|---|---|---|---|---|
| .1 | .818145 | .822562 | .826978 | .831395 | .835811 | .840228 | .844645 | .84906 | .85348 | .85789 |
| .2 | .789985 | .793893 | .797801 | .801709 | .805617 | .809525 | .813434 | .81734 | .82125 | .82516 |
| .3 | .764121 | .767585 | .771049 | .774513 | .777977 | .781441 | .784905 | .78837 | .791833 | .79530 |
| .4 | .740404 | .743479 | .746555 | .74963 | .752705 | .755781 | .758856 | .76193 | .765007 | .76808 |
| .5 | .718687 | .721421 | .724156 | .72689 | .729624 | .732359 | .735093 | .73781 | .740562 | .74330 |
| .6 | .698827 | .701261 | .703696 | .70613 | .708565 | .710999 | .713434 | .71587 | .718303 | .72074 |
| .7 | .680684 | .682854 | .685025 | .687195 | .689366 | .691536 | .693706 | .69588 | .698047 | .70022 |
| .8 | .664126 | .666064 | .668001 | .669939 | .671876 | .673813 | .675751 | .67769 | .679625 | .68156 |
| .9 | .649029 | .650761 | .652492 | .654223 | .655954 | .657686 | .659417 | .66115 | .662879 | .66461 |
| 1 | .635274 | .636823 | .638372 | .63992 | .641469 | .643018 | .644567 | .64612 | .647664 | .64921 |

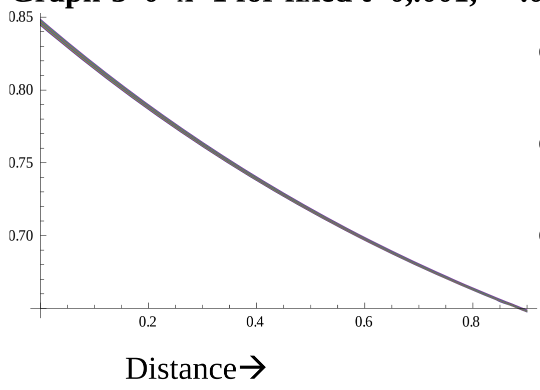
**Graph-1 0<x<1 for fixed t=0,.01,----.**



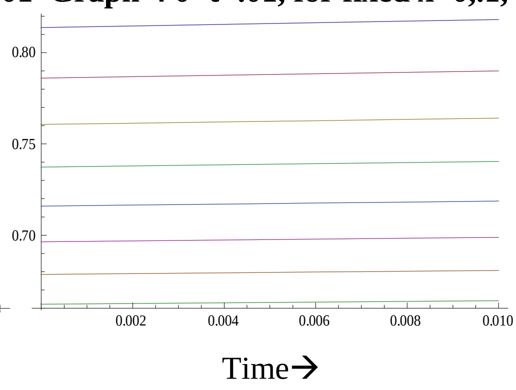

**Graph-2 0<t<.1, for fixed x=0,.1,---1**




**Graph-3 0<x<1 for fixed t=0,.001,----.01**

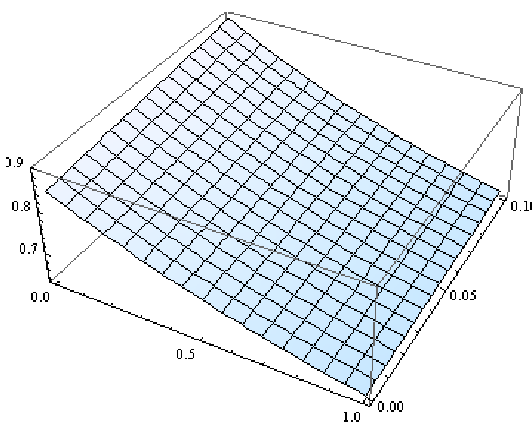


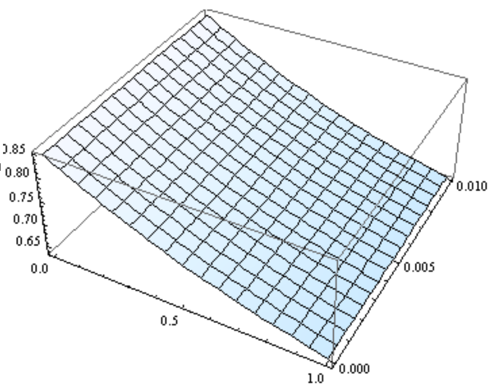

**Graph-4 0<t<.01, for fixed x=0,.1,---1**




**Graph-5 for 0<x<1, 0<t<.1** **Graph-6 for 0<x<1, 0<t<.01**

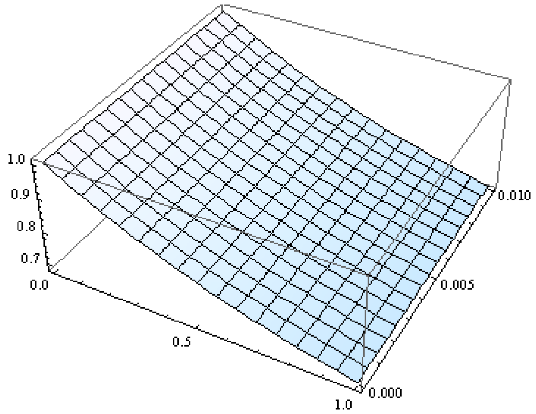
Plot3D[{1/2-Exp[-2x]/32+(3Exp[-x])/8+1/8Exp[-2x]*t+(3Exp[-x]*t)/8},{x,0,1},{t,0,.1}]



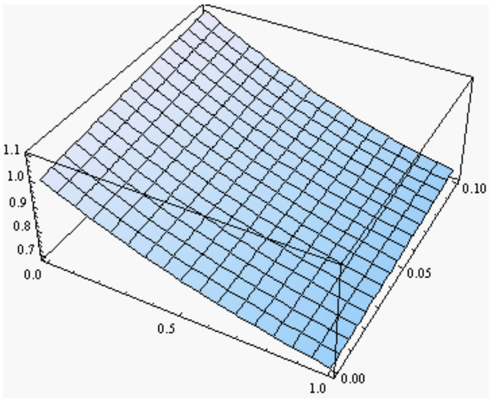



## Exact Solution

**Graph-7 for 0<x<1, 0<t<.01**




**Graph-8 for 0<x<1, 0<t<.1**




## Conclusion Remark

Numerical and Graphical solutions have been developed for predicting the possible concentration of a given dissolved substance in unsteady unidirectional seepage flows through semi-infinite, homogeneous, isotropic porous media subject to the source concentrations that vary negative exponentially with distance. Finally here we make a comparison of numerical solution with the approximate solution which shows Concentration decreases with distance and slightly increases with time. The solution express by equation (21) show that approximate value of concentration (up to two terms) at any x for t >0. The analytical expressions obtained here are useful to the study of salinity intrusion in groundwater, helpful in making quantitative predictions on the possible contamination of groundwater supplies resulting from groundwater movement through buried wastes.

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