Adv. Theor. Appl. Mech., Vol. 3, 2010, no. 8, 385 - 395

Large Unified Symmetries

of Emden Dynamics Systems

Yiping Luo

Department of Physics, Zhejiang Sci-Tech University
Hangzhou 310018, P. R. China
zjstulyp@126.com

Abstract

We propose a new concept of the Large Unified Symmetries and conserved quantities for Emden dynamics systems**.** We work on the Large Unified Symmetries and conserved quantities by the Noether symmetry, the Lie symmetry and the Mei symmetry, including the definition and criterion of the Large Unified Symmetries and conserved quantities deduced from them. The theory of the Large Unified Symmetries for Emden dynamics systems are studied by the relations between the three kinds of symmetries and the three kinds of conserved quantities. The Large Unified Symmetries is a intersection set among the Noether symmetry, the Lie symmetry and the Mei symmetry. Finally, an illustration example is introduced to demonstrate the application of the theory.

Keywords: Large Unified Symmetry, conserved quantity, Emden dynamics system

1 Introduction

In modern mechanics, the symmetry principle is one of the top-level principles. To seek conserved quantities using symmetries of dynamic systems has been a modern development direction[1-7]. Generally speaking, one kind of conserved quantity is presented directly or indirectly by utilizing only one symmetry. Of utmost importance in studying a mechanics system is knowing which physical quantity are conserved, that is, remain constant over time. These quantities might be energy, linear or angular momentum, charge, or other, sometimes less usual, observables, and knowing that such a quantity is conserved can give a wealth of knowledge about the system under consideration.

The methods of symmetry and conserved quantity mainly include the Noether symmetry, the Lie symmetry and the Mei symmetry, and so on. The corresponding conserved quantities are the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity[8-13]. Noether Theorem is an amazing result which lets physicists get conserved quantities from symmetries of the laws of nature. In both classical and quantum physics, Noether Theorem proves to be very powerful because the symmetry of a system are relatively easy to find given the system's Lagrangian. Noether Theorem states that if a system has a particular symmetry, there is a quantity associated with that symmetry that is conserved[14-16]. This result, proved in 1915 by Emmy Noether, was praised by Einstein as a piece of "penetrating mathematical thinking". It's now a standard workhorse in theoretical physics. The symmetry methods for differential equations, originally developed by Sophus Lie in the latter half of the nineteenth century, are highly algorithmic and hence amenable to symbolic computation. For ordinary differential equations and partial differential equations, these include reduction of order through group invariance, integrating factors, construction of special solutions such as similarity solutions, finding conservation laws, equivalence mapping. Lie gave an algorithm to find all infinitesimal generators of point transformations and, more generally, contact transformations admitted by a given differential equation. For a given differential equation, significantly, the basic applications of Lie groups of transformations only require knowledge of the admitted infinitesimal generators. Lie group theory has been developed into a powerful tool to solve differential equations, to classify theory and to establish properties of their solution space[17-23]. Mei studied a new invariance, which means that the dynamical functions in the equation of motion still satisfy the equations of primary form under infinitesimal transformation of groups and is called Mei symmetry. The Mei symmetry can lead to a new conserved quantity directly. References[24-34] gave a new type of conserved quantity, which is also deduced from the Mei symmetry directly but is more general than new conserved quantity.

In this paper, we present the new concept of the Large Unified Symmetries and conserved quantities for Emden dynamics systems by the Noether symmetry, the Lie symmetry and the Mei symmetry. We work on the Large Unified Symmetries and conserved quantities for Emden dynamics systems, including the definition and criterion of the Large Unified Symmetries and the conserved quantities deduced from them. We further study the relations of the Noether symmetry, the Lie symmetry, the Mei symmetry and corresponding conserved quantities for Emden dynamics systems. The theory of the Large Unified Symmetries will play an important role in the fields of modern mechanics.

2 Differential equations of motion for Emden dynamics system

Suppose that configuration of a mechanical system is determined by generalized coordinates $q_s (s=1,\cdots n)$, and its movement is subject to ideal double-side holonomic constraints. The differential equations of motion for Emden dynamics system can be written by

$$\ddot{q}+\frac{2}{t}\dot{q}+q^5=0, \tag{1}$$

and satisfies equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q\,, \tag{2}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is Lagrangian function, $Q = Q(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is generalized non-potential forces. Expanding Eq.(2), we have generalized acceleration as

$$\ddot{q} = \alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\,. \tag{3}$$

Eq.(2) is also written by

$$E(L) = Q\,, \tag{4}$$

where

$$E = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q} \tag{5}$$

is Euler operator.

3 Definition and criterion of Large Unified Symmetries

Choose the infinitesimal transformations of time and coordinates as follow

$$\begin{cases} t^* = t + \varepsilon \xi_0 (t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) = q_s(t) + \varepsilon \xi_s (t, \boldsymbol{q}, \dot{\boldsymbol{q}}), (s = 1, \cdots n) \end{cases} \tag{6}$$

where ε is infinitesimal parameter, ξ_0 and ξ_s are called the generators or the generating functions of the infinitesimal transformations.

Taking the infinitesimal generator as

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}\,, \tag{7}$$

its first expanded vector is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}\,, \tag{8}$$

and its second expanded vector is

$$X^{(2)} = X^{(1)} + (\ddot{\xi}_s - 2\ddot{q}_s \dot{\xi}_0 - \dot{q}_s \ddot{\xi}_0)\frac{\partial}{\partial \ddot{q}_s}\,. \tag{9}$$

Noether identity of Emden systems is also written as

$$L\dot{\xi}_0 + X^{(1)}(L) + Q\left(\xi_s - \dot{q}_s \xi_0\right) + \dot{G}_N = 0\,, \tag{10}$$

where $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a gauge function. The determining equation of Lie symmetry for Emden systems is written by

$$X^{(2)}\left[E(L)\right] = X^{(1)}(Q)\,. \tag{11}$$

The criterion equations of the Mei symmetry of Emden systems can be also expressed by

$$E\left[X^{(1)}(L)\right] = X^{(1)}(Q)\,. \tag{12}$$

Definition For the corresponding Emden system (4), if a symmetry under the transformation (6) is a Noether symmetry as well as a Lie symmetry and is also a Mei symmetry at the same time, then the symmetry is called the Large Unified Symmetry of the system.

Criterion For Emden system (3), if there exists a gauge function $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that the infinitesimal generating functions ξ_0 and ξ_s satisfy the following condition

$$\left[L\dot{\xi}_0 + X^{(1)}(L) + Q\left(\xi_s - \dot{q}_s\xi_0\right) + \dot{G}_N\right]^2 \\ + \left\{X^{(2)}[E(L)] - X^{(1)}(Q)\right\}^2 + \left\{E[X^{(1)}(L)] - X^{(1)}(Q)\right\}^2 = 0, \tag{13}$$

then the symmetry is called the Large Unified Symmetry of Emden systems.

4 Conserved Quantities of Large Unified Symmetries

The Large Unified Symmetries of Emden systems can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity under certain conditions. We have

Proposition 1 For Emden system, the Large Unified Symmetries will lead to the Noether conserved quantity

$$I_N = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_N = \text{const.} \tag{14}$$

Proof: Because the Large Unified Symmetry for Emden system is certainly a Noether symmetry, there exists a gauge function $G_N = G_N(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that the identity (10) holds. According to the Noether theorem, this has the conserved quantity (14). The proof of the theorem is accomplished.

Proposition 2 For Emden system, under the special infinitesimal transformations in which time is it not varied, if there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial \alpha}{\partial \dot{q}_s} + \frac{\bar{d}}{dt}\ln\mu = 0, \tag{15}$$

then the Large Unified Symmetries will lead to the Hojman conserved quantity

$$I_H = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{d}\,\xi_s}{dt}\right) = \text{const}, \tag{16}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha\frac{\partial}{\partial \dot{q}_s}. \tag{17}$$

Proof: Because the Large Unified Symmetry for Emden system is certainly a Lie symmetry, so the determining equation (11) of Lie symmetry holds. From expressions (11) and (15), using Hojman's proof method, Proposition 2 is verified.

Proposition 3 For Emden system, if there exists a function $G_F = G_F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\tilde{X}^{(1)}(L)\frac{\bar{d}\,\xi_0}{dt} + \tilde{X}^{(1)}\left[\tilde{X}^{(1)}(L)\right] + \tilde{X}^{(1)}(Q)\left(\xi_s - \dot{q}_s\xi_0\right) + \frac{\bar{d}\,G_{\mathrm{F}}}{dt} = 0, \tag{18}$$

where

$$\widetilde{X}^{(1)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+\left(\frac{\bar{d}\,\xi_s}{dt}-\dot{q}_s\frac{\bar{d}\,\xi_0}{dt}\right)\frac{\partial}{\partial \dot{q}_s}, \tag{19}$$

then the Large Unified Symmetries will lead to the new conserved quantity

$$I_F=\widetilde{X}^{(1)}(L)\xi_0+\frac{\partial \widetilde{X}^{(1)}(L)}{\partial \dot{q}_s}\left(\xi_s-\dot{q}_s\xi_0\right)+G_F=\text{const.} \tag{20}$$

Proof: Because the Large Unified Symmetry for Emden system is certainly a Mei symmetry, so the criterion equation (12) holds. By using (12) and (18), we can prove that Proposition 3 is true.

5 Illustration example

As an application of Large Unified Symmetries for Emden systems, an illustration example is introduced.

We study the Large Unified Symmetries and conserved quantities for Emden systems . Suppose that the Lagrangian of the system is

$$L=\frac{1}{2}t^2\dot{q}^2. \tag{21}$$

The non-potential generalized forces is

$$Q=-t^2q^5. \tag{22}$$

Choose the infinitesimal generating functions as

$$\xi_0=-t,\ \ \xi=\frac{1}{2}q. \tag{23}$$

It is easy to verify that the generators are the large unified symmetrical for Emden system.

Substituting Eq.(23) into Eq.(10), we can obtain

$$G_N=\frac{1}{6}t^3q^6. \tag{24}$$

From the proposition 1 the Noether conserved quantity gives

$$I_N=\frac{1}{2}\dot{q}^2t^3+\frac{1}{2}q\dot{q}t^2+\frac{1}{6}t^3q^6=\text{const.} \tag{25}$$

The formula (15) gives

$$\frac{\bar{d}}{dt}\ln\mu=0. \tag{26}$$

It has the solution

$$\mu=\dot{q}^{-2}\left(q\ \ -\dot{q}\ t-\frac{1}{4}t^2\right) \tag{27}$$

and the Hojman conserved quantity from the proposition 2 gives

$$I_H = \frac{\dot{q} + \frac{1}{2}t}{q - \dot{q}t - \frac{1}{4}t^2} = \text{const.} \tag{28}$$

On account of

$$\begin{aligned} &\tilde{X}^{(1)}(L) = X^{(1)}(L) = -L, \\ &E[\tilde{X}^{(1)}(L)] = -E(L) = 0, \\ &\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] = -\tilde{X}^{(1)}(L), \end{aligned} \tag{29}$$

the Eq. (18) has solution

$$G_F = 0, \tag{30}$$

then from the proposition 3 we have

$$I_F = \frac{1}{2}t^3\dot{q}^2 + \frac{1}{2}t^2 q\dot{q} + \frac{1}{6}t^3 q^6 = \text{const.} \tag{31}$$

6 Summary and Conclusions

In the paper, we study three kinds of symmetries and their conserved quantities deduced by them for Emden systems. The valuable symmetry and conserved quantity in physics are primitively presented by Noether. Noether studied the invariance of the Hamilton action under the infinitesimal transformations of time and coordinates and brought to light the latent relation between the symmetry and the conserved quantity. The advantage of the Noether theory is that for given symmetry the corresponding conserved quantity can be found by the Noether theorem, and for given conserved quantity the corresponding symmetry can be found by the Noether inverse theorem. Therefore, the Noether theory is an important method searching conserved quantity in the dynamics of constraint systems. Hojman used the Lie symmetry of the differential equations under the infinitesimal transformations in which time is not varied to find directly the conserved quantity constructed without using either Lagrangians or Hamiltonians. We use the Hojman Theorem and its generalizations to study the Hojman conserved quantity for Emden system. The fundamental idea of the Mei symmetry is originated from the work of Mei on the Lagrange system (Mei, 2000). Mei studied the definitions and criteria of his symmetry for the Lagrange system and obtained a kind of new conserved quantity deduced by the form invariance. The new conserved quantity deduced by the Mei symmetry is different from Noether conserved quantity and the Hojman conserved quantity.

Three kinds of symmetries, i.e. the Noether symmetry, the Lie symmetry and the Mei symmetry can lead to directly three kinds of conserved quantities corresponding their symmetries. The three kinds of conserved quantities can be produced indirectly by corresponding their symmetries in some conditions. The Hojman conserved quantity can be deduced by the Noether symmetry for Emden system. In some cases, the Hojman conserved quantity obtained are trivial. Therefore, there is a finding technique. The key to the settlement of the question lies in the choice of the function μ. We study the Lie symmetry and the new conserved quantity deduced by the Lie symmetry for Emden system. The general idea of the method is that if the Lie symmetry is a Mei symmetry, then the new conserved quantity deduced directly by the Mei symmetry will be obtained indirectly by the Lie symmetry. We study the Mei symmetry and the Noether conserved quantity deduced by the Mei symmetry for Emden system. The general idea of the method is that if the Mei symmetry is a Noether symmetry, then the Noether conserved quantity deduced directly by the Noether symmetry can be obtained indirectly by the Mei symmetry. We study the Hojman conserved quantity deduced by the Mei symmetry for Emden system. The general idea of the method is that if the Mei symmetry is a Lie symmetry, then the Hojman conserved quantity deduced directly by the Lie symmetry can be obtained indirectly by the Mei symmetry. The relations between the three kinds of symmetries and the three kinds of conserved quantities for Emden systems are as shown in the Fig.1.

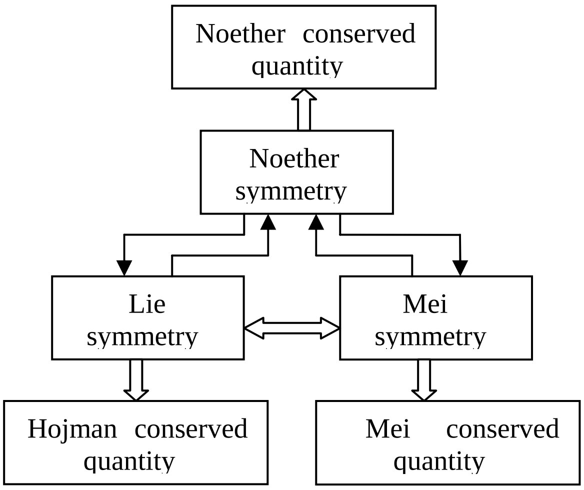

Fig 1: The relations between the three kinds of symmetries and the three kinds of conserved quantities

We present the new concept of Large Unified Symmetries and conserved quantities for Emden systems by the relations between the three kinds of symmetries and the three kinds of conserved quantities. We work on the Large Unified Symmetries and conserved quantities by the Noether symmetry, the Lie symmetry and the Mei symmetry, including the definition and criterion of the Large Unified Symmetries and the conserved quantities deduced from them. The Large Unified Symmetries of Emden systems can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity under certain conditions. The Large Unified Symmetries is a intersection set among the Noether symmetry, the Lie symmetry and the Mei symmetry as shown in the Fig.2.

Because the Noether symmetry is a invariance of Hamilton action under infinitesimal transformations and the Lie symmetry is a invariance of differential equations under group of infinitesimal transformations, and the Mei symmetry means that the forms of the differential equations of motion and the constraint equations keep invariant when the dynamical functions are replaced by the transformed functions under the infinitesimal transformations of group. And the Large Unified Symmetries has not only the invariance of the Hamilton action and the invariance of the differential equations, but has also the invariant forms of the differential equations and the constraint equations. From this point of view, the Large Unified Symmetries is of great significance. The theory about the Large Unified Symmetries for Emden systems will play an important role in the fields of modern theoretical mechanics.

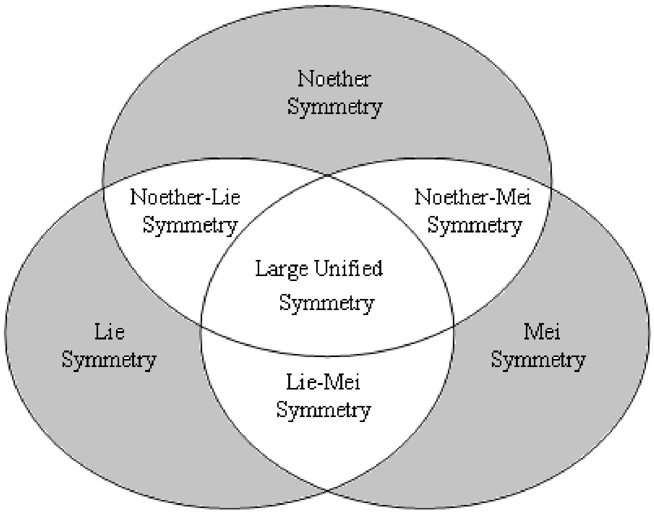

Fig 2: The intersection set of the Large Unified Symmetries

References

[1] P. Hydon. *Symmetry Methods for Ordinary Differential Equations.* Cambridge in press, London,1999.

[2] A. S. Galiullin, G. G. Gafarov, R. P. Malaishka and A. M. Khwan. *Analytical Dynamics of Helmholtz, Birkhoff and Nambu System.* UFN, Moscow,1997(in Russian).

[3] F. X. Mei. *Symmetries and Conserved Quantities of Constrained Mechanical System*. Beijing Institute of Technology Press, Beijing, 2004.

[4] Z. M. Lou. The study of symmetries and conserved quantities for one-dimensional damped-amplified harmonic oscillators. *Acta phys. Sin.,* 57: 1307-1310, 2008.

[5] J. L. Cai. Conformal Invariance and Conserved Quantities of Mei Symmetry for Lagrange Systems. *Acta Phys. Pol. A,* 115: 854-856., 2009.

[6] Y. Zhang, M. Shang and F. X. Mei. Symmetries and conserved quantities for systems of generalized classical mechanics. *Chin. phys. B.,* 9: 401-407, 2000.

[7] W. T. Wan and Y. Chen. A Note on Nonclassical Symmetries of a Class of Nonlinear Partial Differential Equations and Compatibility. *Commun. Theor. Phys.*, 52: 398-402, 2009.

[8] Y. C. Liu, L. L. Xia, X. M. Wang. Unified symmetry of mechanico-electrical systems with nonholonomic constraints of non-Chetaev's type. *Acta phys. Sin.,* 58: 6732-6736, 2009.

[9] Y. Zhang. Conformal invariance and Noether symmetry, Lie symmetry of holonomic mechanical systems in event space. *Chin. phys. B.*, 18: 4636-4642, 2009.

[10] Y. P. Luo and J. L. Fu. Conformal Invariance and Hojman Conserved Quantities for holonomic Systems with Quasi-Coordinates，*Chin. Phys. B.,* 19: 090303:1-6, 2010.

[11] X. W. Chen, Y. H. Zhao, C. Liu. Conformal invariance and conserved quantity for holonomic mechanical systems with variable mass. *Acta phys. Sin.,* 58:5150 -5154, 2009.

[12] Q. B. Hou, Y. C. Li, J. Wang and L. L. Xia. Unified symmetry of the nonholonomic system of non-Chetaev type with unilateral constraints in event space. *Chin. phys. B.*, 16: 1521-1525, 2007.

[13] Y. C. Liu, L. L. Xia, X. M. Wang. Unified symmetry of mechanico-electrical systems with nonholonomic constraints of non-Chetaev's type. *Acta phys. Sin.,* 58: 6732-6736, 2009.

[14] J. L. Fu, W. J. Zhao and Y. Q. Weng. Structure properties and Noether symmetries for super-long elastic slender rod. *Chin. phys. B.*, 17: 2361-2365, 2008.

[15] J. Q. Mei. Noether-Type Symmetries and Associated Conservation Laws of Some Systems of Nonlinear PDEs. *Commun. Theor. Phys.*, 51: 495-498, 2009.

[16] F. X. Mei, H. Wu. *Dynamics of Constrained Mechanical Systems*. Beijing Institute of technology press, Beijing, 2009.

[17] S. Y. Shi, J. L. Fu and Q. Chen. The Lie symmetries and Noether conserved quantities of discrete non-conservative mechanical systems. *Chin. phys. B.*, 17: 385-389, 2008.

[18] J. L. Cai, F. X. Mei. Conformal invariance and conserved quantity of Lagrange systems under Lie point transformation. *Acta phys. Sin.,* 57: 5369-5373, 2008.

[19] Y. P. Luo. Conformal Symmetries and Conserved Quantities of Nonholonomic Systems under Lie Point Transformation. *Chin. Jour. Phys.,* 48: 17 - 26, 2010.

[20] H. B. Zhang. Lie symmetries and conserved quantities of non-holonomic mechanical systems with unilateral Vacco constraints. *Chin. Phys.,* 11: 1 - 4, 2002.

[21] X. W. Chen, C. Liu and F. X. Mei. Conformal invariance and Hojman conserved quantities of first order Lagrange systems. *Chin. phys. B.*, 17: 3180-3184, 2008.

[22] H. B. Zhang, L. Q. Chen, R. W. Liu. First integrals of the discrete nonconservative and nonholonomic systems. *Chin. Phys.,* 14: 238 - 243, 2005.

[23] C. Liu, S. X. Liu, F. X. Mei et al.. Conformal invariance and Hojman conserved quantities of canonical Hamilton systems. *Chin. phys. B.*, 18: 856 - 860, 2009.

[24] N. Ding and J. H. Fanng. A New Type of Mei Adiabatic Invariant Induced by Perturbation to Mei Symmetry of Hamiltonian Systems. *Commun. Theor. Phys.*, 52: 12-16, 2009.

[25] W. K. Ge. Mei symmetry and conserved quantity of a holonomic system. *Acta phys. Sin.,* 57: 6714-6717, 2008.

[26] L. Q. Jia, Y. Y. Zhang, J. C. Cui and S. K. Luo. Structural Equation and Mei Conserved Quantity of Mei Symmetry for Appell Equations in Holonomic Systems with Unilateral Constraints. *Commun. Theor. Phys.*, 55: 572-576, 2009.

[27] T. Pang, J. H. Fang, M. J. Zhang, P. Lin and K. Lu. A New type of conserved quantity deduced from Mei symmetry of nonholonomic systems in terms of quasi-coordinates. *Chin. phys. B.*, 18: 3150-3154, 2009.

[28] J. H. Fang. A kind of conserved quantity of Mei symmetry for Lagrange system. *Acta phys. Sin.*, 58: 3617-3619, 2009.

[29] Y. X. Guo, C. Liu and F. X. Mei. Conformal invariance and conserved quantities of non-conservative Lagrange systems by point transformations. *Chin. phys. B.*, 18: 395 - 399, 2009.

[30] J. H. Fang. A kind of conserved quantity of Mei symmetry for Lagrange system. *Acta. phys. Sin.*, 58: 3617 - 3619, 2009.

[31] J. H. Fang, P. Wang and X. M. Wang. A generalized Mei conserved quantity and Mei symmetry of Birkhoff system. *Chin. phys. B.*, 18: 1312 - 1315, 2009.

[32] J. L. Fu, L. Q. Chen, B. Y. Chen. Noether-type theorem for discrete nonconservative dynamical systems with nonregular lattices. *Sci. China. Ser. G.*, 39: 1320 - 1329, 2009.

[33] Y. Zhang. Routh method of reduction for Birkhoffian systems in the event space. *Chin. phys. B.*, 17: 4365-4368, 2008.

[34] W. K. Ge. Mei symmetry and conserved quantity of a holonomic system. *Acta. Phys. Sin.*, 57: 6714 - 6717, 2008.

Received: June, 2010