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Application of He's Amplitude - Frequency Formulation for Periodic Solution of Nonlinear Oscillators

Jafar Langari*

Islamic Azad University , Quchan Branch, Sama College
PO Box 441, Quchan, Iran

Mehdi Akbarzade

Islamic Azad University , Quchan Branch
Sama College , PO Box 441, Quchan, Iran
mehdiakbarzade@yahoo.com

Abstract

In this paper, a powerful analytical method, called He's amplitude-frequency formulation (HAFF) is used to obtain a periodic solution of nonlinear oscillators differential equation that governs the oscillations of a conservative autonomous system with one degree of freedom.
We illustrate the usefulness and effectiveness of the proposed technique. Some examples are given to illustrate the accuracy and effectiveness of the method. The method can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other science.

Keywords: Nonlinear Oscillators, He's Amplitude-Frequency Formulation, Periodic Solution.

1- Introduction

The study of nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by nonlinear equations. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. There are several methods used to find approximate solutions to nonlinear problems such as modified Lindstedt–Poincare method [9-12], variational iteration method [13], homotopy perturbation method [1-5] and energy balance method [6-8, 16] were used to handle strongly nonlinear systems. He's amplitude-frequency formulation (HAFF) was paid attention recently; it is proven this method is very effective to determine the angular frequencies of strongly nonlinear oscillators with high accuracy [15]. Some examples reveal that even the lowest order approximations are of high accuracy.

2- Basic idea

First we consider the motion of a ball bearing oscillation in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement u. the governing equation, ignoring frictional losses, is [14]:

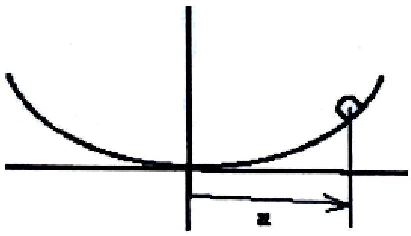

Fig. 1. The motion of a ball bearing oscillation

$$u'' + \varepsilon u^3 = 0\ ,\ u(0) = A\ ,\ u'(0) = 0 \tag{1}$$

According to He's amplitude-frequency formulation [15], we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ where ω is assumed to be the frequency of the nonlinear oscillator Eq (1). Substituting u_1 and u_2 into Eq. (1), we obtain, respectively, the following residuals:

$$R_1 = -A\cos(t) + \varepsilon A^3\cos^3(t) \tag{2}$$

And

$$R_2 = -A\cos(\omega t)\omega^2 + \varepsilon A^3\cos^3(\omega t), \tag{3}$$

If, by chance, u_1 or u_2, is chosen to be the exact solution, then the residual, Eq. (2) or Eq. (3), is vanishing completely. In order to use He's amplitude-frequency formulation, we set:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\, dt = \frac{2\left(-\frac{1}{4}A\pi + \frac{3}{16}\varepsilon A^3\pi\right)}{\pi}\ , T_1 = 2\pi \tag{4}$$

And:

(5)

$$R_{22}=\frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\,dt=-\frac{1}{8}\frac{A\left(-3\varepsilon A^2\pi+4\omega^2\pi\right)}{\pi}\ ,\ T_2=\frac{2\pi}{\omega}$$

Applying He's frequency-amplitude formulation [15], we have:

$$\omega^2=\frac{\omega_1{}^2R_{22}-\omega_2{}^2R_{11}}{R_{22}-R_{11}} \tag{6}$$

Where:

$$\omega_1=1\ ,\ \omega_2=\omega \tag{7}$$

We, therefore, obtain:

$$\omega^2=\frac{3}{4}A^2\varepsilon \tag{8}$$

The first order approximate solution is obtained, which reads:

$$\omega=\sqrt{\frac{3}{4}A^2\varepsilon} \tag{9}$$

Its period can be written in the form:

$$T=\frac{2\pi}{\sqrt{\frac{3}{4}A^2\varepsilon}}=\frac{4\pi}{\sqrt{3\varepsilon}}A^{-1}=7.2552A^{-1}\varepsilon^{-1} \tag{10}$$

The exact period [14] is $T=7.4163A^{-1}\varepsilon^{-1}$. Therefore, it can be easily proved that the maximal relative error is less than 2.17%.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly.

3- Applications

In order to assess the advantages and the accuracy of the He's amplitude-frequency formulation (HAFF); we will consider the following examples.

3.1- Example 1

We consider the the well-known Duffing equation [14]:

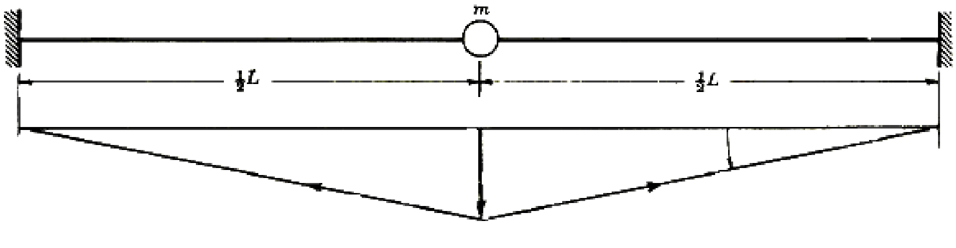

Fig. 2. The physical model of Duffing equation

$$u'' + u + \varepsilon u^3 = 0 \tag{11}$$

With initial condition of: $u(0) = A$, $u'(0) = 0$

According to He's amplitude-frequency formulation [15], we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ where ω is assumed to be the frequency of the nonlinear oscillator Eq (11). Substituting u_1 and u_2 into Eq. (11), we obtain, respectively, the following residuals:

$$R_1 = \varepsilon A^3 \cos^3(t) \tag{12}$$

And

$$R_2 = -A\cos(\omega t)\omega^2 + A\cos(\omega t) + \varepsilon A^3\cos^3(\omega t), \tag{13}$$

In order to use He's amplitude-frequency formulation, we set:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\,dt = \frac{3}{8}\varepsilon A^3 \ , \ T_1 = 2\pi \tag{14}$$

And:

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\,dt = \frac{1}{8}\frac{A\left(3\varepsilon A^2\pi - 4\omega^2\pi + 4\pi\right)}{\pi} \ , \ T_2 = \frac{2\pi}{\omega} \tag{15}$$

Applying He's frequency-amplitude formulation [15], we have:

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}} \tag{16}$$

Where:

$$\omega_1 = 1 \ , \ \omega_2 = \omega \tag{17}$$

We, therefore, obtain:

$$\omega^2 = 1 + \frac{3}{4}A^2\varepsilon \tag{18}$$

The first order approximate solution is obtained, which reads:

$$\omega = \sqrt{1 + \frac{3}{4}A^2\varepsilon} \tag{19}$$

What is rather surprising about the remarkable range of validity of (19) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty}\frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi}\int_0^{\pi/2}\frac{dx}{\sqrt{1-0.5\sin^2 x}} = 0.9294 \tag{20}$$

The lowest order approximation given by (19) is actually within 7.6% of the exact frequency regardless of the magnitude of εA^2 .

3.2- Example 2

We consider the quadratic nonlinear oscillator [17]:

$$u'' + u + \varepsilon u^2 = 0 \tag{21}$$

With initial condition of: $u(0) = A$, $u'(0) = 0$

According to He's amplitude-frequency formulation [15], we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ where ω is assumed to be the frequency of the nonlinear oscillator Eq. (21). Substituting u_1 and u_2 into Eq. (21), we obtain, respectively, the following residuals:

$$R_1 = \varepsilon A^2\cos^2(t) \tag{22}$$

And

$$R_2 = -A\cos(\omega t)\omega^2 + A\cos(\omega t) + \varepsilon A^2\cos^2(\omega t), \tag{23}$$

In order to use He's amplitude-frequency formulation, we set:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\,dt = \frac{4}{3}\frac{\varepsilon A^2}{\pi}\ ,\ T_1 = 2\pi \tag{24}$$

And:

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\,dt = \frac{1}{6}\frac{A\left(8\varepsilon A - 3\omega^2\pi + 3\pi\right)}{\pi}\ ,\ T_2 = \frac{2\pi}{\omega} \tag{25}$$

Applying He's frequency-amplitude formulation [15], we have:

$$\omega^2 = \frac{\omega_1^{\,2}R_{22} - \omega_2^{\,2}R_{11}}{R_{22} - R_{11}} \tag{26}$$

Where:

$$\omega_1 = 1\ ,\ \omega_2 = \omega \tag{27}$$

We, therefore, obtain:

$$\omega^2 = 1 + \frac{8}{3\pi}\varepsilon A \tag{28}$$

The first order approximate solution is obtained, which reads:

$$\omega = \sqrt{1 + \frac{8}{3\pi}\varepsilon A} \tag{29}$$

Application of the Lindstedt–Poincare´ method to Eq. (21) gives the following second approximation [27]:

$$u(t,\varepsilon) = A\cos\omega_2 t + \varepsilon\left(\frac{A^2}{6}\right)(-3 + 2\cos\omega_2 t + \cos 2\omega_2 t) + \varepsilon^2\left(\frac{A^3}{3}\right)\left[-1 + \frac{29}{48}\cos\omega_2 t + \frac{1}{3}\cos 2\omega_2 t + \frac{1}{16}\cos 3\omega_2 t\right] + O(\varepsilon^3) \tag{30}$$

And:

$$\omega_2 = 1 - \varepsilon^2\left(\frac{5A^2}{12}\right) + O(\varepsilon^3)\ ,\ 0 < A << 1 \tag{31}$$

The Lindstedt–Poincare´ method usually applies to weakly nonlinear oscillator problems [9-12]. The method of He's frequency-amplitude formulation is capable of producing analytical approximation to the solution to the nonlinear system, valid even for the case where the nonlinear terms are not ''small''. And also in order to compare with harmonic balance result we write [17]:

$$\omega_{HB} = \sqrt{1 + \frac{8}{3\pi}\varepsilon A} \qquad (32)$$

3.3- Example 3

We consider the quadratic and cubic nonlinear oscillator [18]:

$$u'' + u + \varepsilon u^2 + u^3 = 0 \qquad (33)$$

With initial condition of: $u(0) = A$, $u'(0) = 0$

According to He's amplitude-frequency formulation [15], we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ where ω is assumed to be the frequency of the nonlinear oscillator Eq. (33). Substituting u_1 and u_2 into Eq. (33), we obtain, respectively, the following residuals:

$$R_1 = \varepsilon A^2\cos^2(t) + A^3\cos^3(t) \qquad (34)$$

And

$$R_2 = -A\cos(\omega t)\omega^2 + A\cos(\omega t) + \varepsilon A^2\cos^2(\omega t) + A^3\cos^3(\omega t), \qquad (35)$$

In order to use He's amplitude-frequency formulation, we set:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\, dt = \frac{1}{\pi}\left(\frac{4}{3}\varepsilon A^2 + \frac{6}{16}A^3\pi\right), \; T_1 = 2\pi \qquad (36)$$

And:

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\, dt = \frac{A}{24\pi}\left(32\varepsilon A - 12\omega^2\pi + 9A^2\pi + 12\pi\right), \; T_2 = \frac{2\pi}{\omega} \qquad (37)$$

Applying He's frequency-amplitude formulation [15], we have:

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}} \qquad (38)$$

Where:

$$\omega_1 = 1, \; \omega_2 = \omega \qquad (39)$$

We, therefore, obtain:

$$\omega^2 = 1 + \frac{8}{3\pi}\varepsilon A + \frac{3}{4}A^2 \qquad (40)$$

The first order approximate solution is obtained, which reads:

$$\omega = \sqrt{1 + \frac{8}{3\pi}\varepsilon A + \frac{3}{4}A^2} \qquad (41)$$

Application of the Lindstedt–Poincare´ method to Eq. (33) gives the following second approximation [14]:

$$u(t,\varepsilon) = A\cos\omega_2 t + \varepsilon\left(\frac{A^2}{6}\right)(-3 + 2\cos\omega_2 t + \cos 2\omega_2 t) + \qquad (42)$$

$$\left(\frac{A^3}{3}\right)\left[-\varepsilon^2 + \left(\frac{174\varepsilon^2 - 27}{288}\right)\cos\omega_2 t + \frac{\varepsilon^2}{3}\cos 2\omega_2 t + \left(\frac{2\varepsilon^2 + 3}{32}\right)\cos 3\omega_2 t\right] + O(\varepsilon^3)$$

And:

$$\omega_2 = 1 + A^2\left(\frac{9-10\varepsilon^2}{24}\right) + O(\varepsilon^3)\ ,\ 0 < A << 1 \tag{43}$$

We see that the first approximations obtained in this paper are more accurate than Lindstedt–Poincare´ results for large amplitudes. And also in order to compare with harmonic balance result we write [18]:

$$\omega_{HB} = \sqrt{1 + \frac{8}{3\pi}\varepsilon A + \frac{3}{4}A^2} \tag{44}$$

4- Conclusions

This paper has proposed a new method for solving accurate analytical approximations to strong nonlinear oscillations.

The solution procedure of He's amplitude-frequency formulation (HAFF) is of deceptive simplicity and the insightful solutions obtained are of high accuracy even for the one-order approximation. The method, which is proved to be a powerful mathematical tool to the search for natural frequencies of nonlinear oscillators, can be easily extended to any nonlinear equation, we think that the method has a great potential and can be applied to other strongly nonlinear equations.

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