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LTB Gravity Minimally Coupled to Scalar Field: a Cosmology Admitting Short Time Inflation and Large Time Constant Velocity Expansion

Antonio Zecca

GNFM, Gruppo Nazionale di Fisica Matematica of the INdAM, Italy
INFN, Istituto Nazionale di Fisica Nucleare Sezione di Milano, Italy
Dipartimento di Fisica dell' Universita' degli Studi di Milano, Retired

Abstract

The study of the minimal coupling of scalar field with gravity in a general comoving spherically symmetric system is reconsidered. Spherical symmetry implies the independence of the field on the angular variables and the equivalence of the consistency of Einstein equation condition to the vanishing of the divergence of the energy momentum tensor of the scalar field. The explicit system of equations is reduced to the solution of a Kepler like equation coupled to the scalar field equation. The equations are studied in the special case in which the scalar field is massless and depends only on the time coordinate. As a consequence the complexity of the system of equations strongly reduces. The equations can be finally integrated by variable separation. By using the asymptotic behaviors of the explicit solution for the physical radius one can see that the the induced cosmological model shows an inflation at short time and a constant expansion velocity at large time .

Keywords: LTB space-time - Scalar field minimally coupled to Einstein field equation - Inflation - Expansion

1 Introduction

The study of the coupling of scalar and gravitational field is of interest in General Relativity under many aspects. They run from mathematics to physical applications. Self gravitational interaction of matter fields has been widely studied on mathematical ground in case of massless field in spherically symmetric space-time [3, 4]. The problem is also relevant for the formulation of a cosmological model. It plays indeed a fundamental role to explain the dynamics of inflation (e. g., [6]), the description of gravitational collapse and black hole and singularity formation (e. g., [8, 2, 13]). Gravity, non minimally coupled to scalar field, has been considered in connection with inflation, accelerated expansion of the Universe and dark matter production [5, 9].

The interaction of gravity with minimally coupled massive scalar field, was previously studied, to different extent, in Lemaître-Tolman-Bondi (TLB) space-time, that represents the most general spherically symmetric comoving system. That model is of interest in the formulation of inhomogeneous cosmological model [7].

Once made explicit the scheme gives rise to a set of coupled non linear partial differential equations.

On account of the spherical symmetry the field ϕ is allowed to depend only on the radial and time coordinates. The validity of the scalar field equation implies the consistency of the Einstein equation, namely the requirement that $T_{\mu\nu}(\phi)$ has zero divergence.

The study of the explicit equations was previously performed in the static case [12]. It greatly simplifies if the scalar field depends on only one of the space time coordinates r, t [14].

It is possible however to go further in the development of the general scheme. The solution can be reduced to the solution of a generalized Kepler like cosmological equation coupled to the scalar field. The equations result however to have an involved structure and the very problem is their decoupling [15].

The scheme simplifies in the special case of the Robertson Walker space-time [15]. In that case, it can be reported to the solution of separated differential equations, non linear in the fields and in the physical "radius" of the Universe as discussed in a special case.

More generally, the scheme greater simplifies in case the scalar field depends on one only space time coordinate.

In the present paper attention is focused to the case of massless scalar field that depends only on the time coordinate. I such case the coupled Kepler-like and scalar field equations can be disentangled. A closed equation in the physical radius can be obtained that can be integrated by variable separation. By asymptotic analysis of the solution one can see that the corresponding cosmological model admits of an inflation at short time and an expansion at constant velocity at large time.

2 Formulation of the scheme and basic results

The object is to study, in a General Relativity context, the interaction of a scalar field with gravity. In terms of equations the (minimally coupled) scalar field equation of the scalar field $\phi(x^\mu)$ is coupled to the Einstein field equation whose source is the energy momentum tensor $T_{\mu\nu}$ of ϕ:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \qquad (\kappa = 8\pi G/c^4) \tag{1}$$

$$\nabla^\alpha\nabla_\alpha\phi + m_o^2\phi = 0 \tag{2}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - (1/2)g_{\mu\nu}[\partial^\alpha\phi\partial_\alpha\phi - m_0^2\phi^2] \tag{3}$$

(m_0 the mass of the field). By contracting the indexes, the eq. (1) can be recast into the form:

$$R_{\mu\nu} = \kappa(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_0^2\phi^2 g_{\mu\nu}) \tag{4}$$

Therefore the problem can be equivalently reduced to the study of the coupled equation (2) and (4).

The interaction scheme is now studied in the framework of the Lemaître Tolman Bondi (LTB) space time, namely within the general comoving spherically symmetric metric tensor $g_{\mu\nu}$ defined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^{\gamma(r,t)}dr^2 - y^2(r,t)\Big(d\theta^2 + \sin\theta^2 d\varphi^2\Big) \tag{5}$$

A first consequence of the spherical symmetry assumption is that $\phi(x^\mu)$ cannot depend on the angular variables. Indeed, from the detailed expression of the Ricci tensor $R_{\mu\nu}$ given in (7)-(10), one has that (see for example [14]) if $\mu \neq \nu$, then $R_{\mu\nu} = 0$ for every $(\mu,\nu) \neq (t,r)$. Hence

$$\partial_t\phi\,\partial_\theta\phi = \partial_\varphi\phi\partial_r\phi = 0 \quad \rightarrow \quad \partial_\theta\phi = \partial_\varphi\phi = 0 \quad \rightarrow \quad \phi = \phi(t,r) \tag{6}$$

Accordingly, by expanding the calculations, the equations (1), (2) become ($\dot{A} = \partial_t A, A' = \partial_r A$):

$$R_{tt} \equiv \frac{2\ddot{\gamma} + \dot{\gamma}^2}{4} + 2\frac{\ddot{y}}{y} = \kappa(\dot{\phi}^2 - \frac{1}{2}m_0^2\phi^2) \tag{7}$$

$$R_{rr} \equiv \frac{2yy'' - y'\gamma' y}{y^2} - e^\gamma(\frac{\dot{\gamma}^2}{4} + \dot{\gamma}\frac{\dot{y}}{y} + \frac{\ddot{\gamma}}{2}) = \kappa(\phi'^2 + \frac{1}{2}e^\gamma m_0^2\phi^2) \tag{8}$$

$$R_{\theta\theta} \equiv -1 + \frac{e^{-\gamma}}{2}[2yy'' + 2y'^2 - yy'\gamma'] - y\ddot{y} - \dot{y}^2 - \frac{y\dot{y}\dot{\gamma}}{2} = \frac{\kappa}{2}\,m_0^2\phi^2\,y^2 \tag{9}$$

$$R_{tr} \equiv 2\frac{\dot{y}'}{y} - \dot{\gamma}\frac{y'}{y} = \kappa\phi'\dot{\phi} \tag{10}$$

$$F(\phi) \equiv \ddot{\phi} - e^{-\gamma}\phi'' + e^{-\gamma}(\frac{\gamma'}{2} - 2\frac{y'}{y})\phi' + (\frac{\dot{\gamma}}{2} + 2\frac{\dot{y}}{y})\dot{\phi} + m_0^2\phi = 0 \tag{11}$$

(The $R_{\phi\phi}$ case reproduces the $R_{\theta\theta}$ case because $R_{\phi\phi} = \sin^2\theta\ R_{\theta\theta}$). The form (11) of the scalar field equation follows, e. g., by specializing to the present case the results in [11].

Another consequence of spherical symmetry assumption (5) is that the consistency condition of the Einstein field equation, namely $\nabla^\mu T_{\mu\nu}(\phi) = 0$, is automatically verified.

By direct computation one has indeed [14]: $\nabla^\mu T_{\mu\theta} = \nabla^\mu T_{\mu\varphi} = 0$. Moreover $\nabla^\mu T_{\mu t} = \dot{\phi}F(\phi) = 0$ and $\nabla^\mu T_{\mu r} = \phi'\ F(\phi) = 0$ by (11).

There are further general aspects of the scheme that can be considered before discussing the solution.

3 Main developments

Note first that eq. (10) can be integrated exactly to obtain (see e.g., [15]):

$$e^\gamma = \frac{y'^2}{1+2E}\exp\left[-k\int_0^t dt\,\frac{y(r,t)}{y'(r,t)}\dot{\phi}(r,t)\phi'(r,t)\right] \tag{12}$$

where the arbitrary radial integration function has been given the form $1 + 2E(r)$ in analogy to the LTB cosmological model. By using (12) in (7) and by taking into account the equations (8), (9) in the resulting equation, one is finally left with

$$\frac{\dot{y}^2}{2} + \frac{M(r,t)}{y} + \frac{1}{2} = \frac{1}{2}e^{-\gamma}y'^2 + \frac{k}{4}y^2(\dot{\phi}^2 + \phi'^2e^{-\gamma} - m_0^2\phi^2) \tag{13}$$

$$M(r,t) = \int_0^r dr\left[ky^2y'\left(\dot{\phi}^2 - \frac{m_0^2}{2}\phi^2\right) - \frac{k^2}{4}\frac{y^4}{y'}\phi'^2\dot{\phi}^2 + \frac{k}{2}\frac{y^2}{y'}\partial_t(\dot{\phi}\phi'yy')\right] \tag{14}$$

The problem of the solution of the scheme is therefore shifted to find the simultaneous solution of (11) and (13) with (14). The equation (13) is still a very involved equation. Formally it can be interpreted as a generalized Kepler-like equation for a particle of zero mass subjected to the gravitational field of a mass $M(r,t)$ (the action of the scalar field included) and the scalar field itself. The scheme was already studied in [15, 16, 17] in different way.

4 Purely time dependent massless scalar field

The object is now to study the previous scheme under the assumptions:

$$\phi = \phi(t), \qquad m_0 = 0 \tag{15}$$

Accordingly the equation (12), (14) simplify to

$$e^{\gamma} = \frac{y'^2}{1+2E}, \quad (2E+1>0) \tag{16}$$

$$M(r,t) = \kappa\, \dot\phi^2\, \frac{y^3}{3} \quad \Longleftrightarrow \quad \frac{3M}{y} = \kappa\, \dot\phi^2\, y^2 \tag{17}$$

while the Kepler-like equation can be given the form:

$$\frac{1}{2}\dot y^2 + \kappa\, \frac{\dot\phi^2}{12}\, y^2 = E \tag{18}$$

The scalar field equation reduces to $\ddot\phi + (\dot\gamma/2 + 2\dot y/y)\dot\phi = 0$. It can be integrated in the form

$$\dot\phi^2 = \frac{1}{y^4}\, e^{-\gamma}\, f(r) \tag{19}$$

with $f(r)$ arbitrary radial integration function.

Finally the Kepler-like equation can be reported to the form:

$$\frac{1}{2}\dot y^2 + \frac{\kappa}{12}\, \frac{(1+2E)}{y^2 y'^2}\, f(r) = E \tag{20}$$

that, by further setting

$$y(t,r) = s(r)\, \tau(t), \qquad f(r) = \frac{s'^2 s^4}{1+2E} \tag{21}$$

gives:

$$\frac{\dot\tau^2(t)}{2} s^2(r) + \frac{\kappa}{12}\, \frac{1}{\tau^4(t)}\, s^2(r) = E(r) \tag{22}$$

The equation (22) is easily separable. One has :

$$s^2 = 2\, E/C \qquad\qquad s = \pm\sqrt{2E/C} \tag{23}$$

$$\dot\tau^2(t) + \frac{\kappa}{6}\, \frac{1}{\tau^4(t)} = C \tag{24}$$

C the separation constant. The equation (24) integrates by variable separation:

$$t - t_0 = \pm \int d\tau \frac{\tau^2}{\sqrt{a\tau^4 + b}}, \qquad a = C, \quad b = -\kappa/6 \tag{25}$$

$$= \pm\frac{1}{2} \int dx \sqrt{\frac{x}{ax^2+b}}, \qquad \tau^2 = x \tag{26}$$

$$= \pm\frac{1}{3}\frac{x^{3/2}}{b^{1/2}}\; {}_2F_1\Big(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; -\frac{a}{b}x^2\Big) \qquad [10] \tag{27}$$

(For the definition of the hypergeometric function ${}_2F_1$ see [1]).

It is of interest the behavior of the solution for small and large t. Directly from (26) one has for $\tau \to 0$

$$t - t_0 \;\cong\; \frac{1}{3\sqrt{b}}\tau^3, \qquad \tau \cong\; (3\sqrt{b})^{1/3}\,(t - t_0)^{1/3} \tag{28}$$

If $\tau \to \infty$, let's consider the general asymptotic behavior of ${}_2F_1$ [1]:

$${}_2F_1(a', b'; c'; z) \overset{z\to\infty}{\longrightarrow} \frac{\Gamma(c')\Gamma(b'-a')}{\Gamma(b')\Gamma(c'-a')}(-z)^{-a'} + \frac{\Gamma(c')\Gamma(a'-b')}{\Gamma(a')\Gamma(c'-b')}(-z)^{-b'} \tag{29}$$

When applied to (27) in case of large τ it gives:

$$t - t_0 \quad \overset{\tau\to\infty}{\longrightarrow} \quad \pm\frac{1}{3b^{1/2}}x^{3/2}\left[\gamma_1\Big(\frac{a}{b}\Big)^{-\frac{1}{2}}(x^2)^{-\frac{1}{2}} + \gamma_2\Big(\frac{a}{b}\Big)^{-\frac{3}{4}}(x^2)^{-\frac{3}{4}}\right] \tag{30}$$

$$\overset{\tau\to\infty}{\longrightarrow} \quad \pm\frac{1}{3}a^{-\frac{1}{2}}\gamma_1\,\tau \tag{31}$$

$$\Big(\gamma_1 = \frac{\Gamma(\frac{7}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})}, \qquad \gamma_2 = \frac{\Gamma(\frac{7}{4})\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{2})\Gamma(1)}\Big)$$

If the present scheme is interpreted as a cosmological model, the physical radius y behaves like $y = s(r)\tau(t) \sim (t - t_0)^{1/3}$ for small $t - t_0$ according (28). There is therefore an inflation at $t = t_0$.

If instead large time is considered, from (31) one has

$$\tau(t) = 3\; a^{\frac{1}{2}}\,\gamma_1^{-1}(t - t_0) \tag{32}$$

that is, there is an expansion of the Universe at constant velocity at large time.

5 Comments

The problem of the minimal coupling of the LTB gravity and the scalar field has been reconsidered here also in view of a cosmological interpretation. The coupled equations to solve are highly non linear and involved, even after a general elaboration and reduction.

Mathematically the solution of the scheme can be explicitly obtained in case the scalar field is massless and depends only on the space time coordinate. In such case the equations strongly simplify and one can finally report the problem to the solution of a single non linear equation closed in the physical radius y. Such equation result to be separable and the separated equations are integrated.

Physically, by considering the asymptotic behavior of the determined solution, one finds that the cosmological interpretation of the results leads to the description of a Universe with an inflation at short time and a constant velocity expansion at large time.

It is open the problem of the solution of the above scheme by relaxing requirements done on the scalar field.

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