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"Dirac Equation in Space-Time with Torsion": Improvements and Further Comments

Antonio Zecca

Dipartimento di Fisica dell' Universita' degli Studi di Milano
Via Celoria, 16 20133 Milano
GNFM *Gruppo Nazionale per la Fisica Matematica* of the INdAM, Italy

Abstract

The derivation of Dirac equation previously studied in a curved space-time with torsion is reconsidered. As in the original paper the total action, sum of the Einstein Hilbert Cartan and of the Dirac action, is considered. The Dirac equation fallows, with improvements, by varying with respect to the Dirac spinor and to the torsion field, Variation with respect to the gravitational field is not considered.

The equation is translated into the language of the 2 spinor formalism of Newman and Penrose with many improvement and clarifications. Contrarily to the original result, here there are at least two possible forms of the final equation. Both form have the same torsion dependent part. Instead the torsion free part in one case leads to Chandrasekhar like formulation (as in the original paper), while in the other case to the Penrose - Rindler's one. The torsion induced non linearity is represented by the interaction of the particle with its own current that remains conserved.

Keywords: Dirac equation, tensors, spinors, torsion

1 Introduction

The extension of Dirac equation to include torsion is an argument of interest from both a mathematical and physical point of view. As it appears it has the advantage of introducing new degree of freedom in the theory suitable to study new physical situation (some basic studies are , e., g. [5, 6, 7, 8, 9]).

The price to pay is the introduction of new non linear terms in the field equations. Such aspect raised however new interest in particle physics and in particular in applications of Dirac equation with torsion (e., g. [15], [3], [1], [25]. At the present the consideration of torsion is an argument generally enclosed in quantum field theory (e. g. [2]).

The Ref. [17] was then dedicated to have the counterpart in the language of Newman and Penrose formalism [11], of the Dirac equation with torsion a subject that seemed to lack in the literature.

In that paper, the Dirac equation with torsion was considered in a 4-dimensional space-time with affine connection. The spin 1/2 equation has been first derived, in the coordinate formalism, by varying the total action S, sum of the Dirac S_D and the Einstein-Hilbert-Cartan S_{EHC} action, with respect to the Dirac and torsion fields. The result was transformed into the 2 dimensional spinor formalism by the standard representation of the Dirac matrices by σ-matrices based on the null tetrad frames. The result of Ref. [17] is a Dirac like equation with torsion whose expanded torsion free part has a Chandrasekhar like form [4]. The effect of torsion amounts to an interaction of the particle with its own (Dirac) current.

In the present paper that problem is reconsidered and the argument developed in the same line of Ref. [17]. However many results are improved by a more precise direct or alternative way and under more clear assumptions. A central new result is that the form of the torsion free part of the final expanded equations may be of Penrose-Rindler or of Chandrasekhar type. That depends on how the σ-matrices are chosen to act on Dirac two spinors. Instead the torsion induced terms are not affected by that choice.

2 Space-time with torsion

In the following the space-time is assumed to be a 4-dimensional Lorentz manifold (M, g) endowed by an affine connection ∇ (we refer to Ref. [10] for notations and mathematical conventions). The affine connection ∇, of affine coefficients $\Gamma^{\lambda}{}_{\mu\nu}$, is required to be a *metric compatible* connection $\nabla_\lambda g_{\mu\nu} = 0$ that is

$$\partial g_{\mu\nu} - \Gamma^{\kappa}{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^{\kappa}{}_{\lambda\nu} g_{\mu\kappa} = 0 \tag{1}$$

The Levi-Civita connection $\widetilde{\nabla}$ that acts on the same space-time, is the one defined by the Christoffel coefficients $\{{}^{\kappa}_{\lambda\nu}\} = \frac{1}{2} g^{k\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$. The relation between ∇ and $\widetilde{\nabla}$ is expressed by

$$\Gamma^{\kappa}{}_{\mu\nu} = \{{}^{\kappa}_{\mu\nu}\} + K^{\kappa}{}_{\mu\nu} \tag{2}$$

The contorsion tensor $K^{\kappa}_{\ \mu\nu}$ is defined through the torsion tensor $T^{\kappa}_{\ \mu\nu} = \Gamma^{\kappa}_{\ \mu\nu} - T^{\kappa}_{\ \nu\mu}$ by means of $K^{\kappa}_{\ \mu\nu} \equiv \frac{1}{2}(T^{\kappa}_{\ \mu\nu} + T_{\mu\ \nu}^{\ \kappa} + T_{\nu\ \mu}^{\ \kappa})$. Since both $\widetilde{\nabla}$ and ∇ are metric compatible, from (2) one has $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$.

On account of the decomposition (2), the Rieman curvature tensor

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}_{\ \nu\lambda} - \partial_{\nu}\Gamma^{\kappa}_{\ \mu\lambda} + \Gamma^{\eta}_{\ \nu\lambda}\Gamma^{\kappa}_{\ \mu\eta} - \Gamma^{\eta}_{\ \mu\lambda}\Gamma^{\kappa}_{\ \nu\eta} \tag{3}$$

and the scalar curvature $R = g^{\mu\nu}R^{\lambda}_{\mu\lambda\nu}$ can be separated into a part expressed only by the Christoffel symbols and a part containing the contorsion.

In view of the following purposes it is convenient decompose also the contorsion in the form (see, e. g. [1] and references therein):

$$K_{\alpha\mu\nu} = \frac{1}{3}(g_{\alpha\mu}\tau_{\nu} - g_{\nu\mu}\tau_{\alpha}) + \frac{1}{2}\mathcal{A}^{\sigma}\epsilon_{\sigma\alpha\mu\nu} + U_{\alpha\mu\nu}, \tag{4}$$

$$\tau_{\mu} = g^{\alpha\beta}K_{\alpha\beta\mu}, \qquad \mathcal{A}^{\sigma} = \frac{1}{3}\epsilon^{\sigma\alpha\beta\mu}K_{\alpha\beta\mu}, \tag{5}$$

$$U_{\alpha\mu\nu} = -U_{\nu\mu\alpha}, \qquad g^{\alpha\mu}U_{\alpha\mu\nu} = 0, \qquad \epsilon^{\sigma\alpha\mu\nu}U_{\alpha\mu\nu} = 0 \tag{6}$$

The term $U_{\alpha\mu\nu}$ is infact determined by eq. (4) and it has By using eqs. (2), (4-6) into the expression (4) one gets, with some rearrangements

$$R = \widetilde{R} - \frac{2}{\sqrt{g}}\partial_{\kappa}(\sqrt{g}\tau^{\kappa}) - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu} \tag{7}$$

where $g = |det\ g_{\mu\nu}|$ and $\widetilde{R}$ is the part expressed in terms of the Chrisoffel symbols alone not containing the contorsion tensor. The field equations can then be obtained by varying the Einstein-Hilbert-Cartan action

$$S_{EHC} = \int d^4x\sqrt{g}(\widetilde{R} - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu}) \tag{8}$$

with respect to the metric and the contortion tensor fields. In (8) the divergence term has been neglected because no variations of the boundary will be considered.

3 Dirac equation with torsion

The description of a Dirac spinor ψ in the 4-dimensional Lorentz manifold M can be done in general by a well known scalar action whose Lagrangian is a scalar both under coordinated change and Lorentz rotations. To do this let e_a^{μ} be a local reference frame (tetrad), with associated inverse matrix $e^b_{\ \nu}$, defined by [10, 16]

$$g_{\mu\nu} = e^a_{\ \mu}e^b_{\ \nu}\eta_{ab}; \qquad \eta_{ab} = e_a^{\mu}e_b^{\nu}g_{\mu\nu} \tag{9}$$

$\eta_{ab} = \eta^{ab} = diag(1,-1,-1,-1)$ the Minkowski metric. Tetrad indices, are denoted by latin letters, coordinate indices by greek letters. Given the Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ the γ^μ matrices defined by $\gamma^\mu = e_a^\mu \gamma^a$ satisfy the relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The mentioned action is given by (e., g. [10, 16]):

$$S_D = \int d^4x \sqrt{g}\, \overline{\psi}[i\gamma^a e_a^\mu(\partial_\mu + \Omega_\mu) + m]\psi \tag{10}$$

As usual $\overline{\psi} = \psi^+\gamma^o$. The term involving the spin connection Ω_μ is defined by

$$\Omega_\mu = -\frac{1}{2}\, i\, \Gamma^{a\ \ b}_{\ \mu}\Sigma_{ab} \tag{11}$$

with $\Sigma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$, $\Gamma^{a\ \ b}_{\ \mu} = e^a_\mu \nabla_\mu e^a_\mu$ and ∇, Γ are the affine connection and the affine coefficients. One has:

$$\gamma^\mu\Omega_\mu \quad = \frac{1}{8} e_a^\mu \gamma^a e_b^\nu \nabla_\mu e_c^\nu [\gamma^b, \gamma^c] \tag{12}$$

$$= \frac{1}{8} e_a^\mu e_b^\nu (\nabla_\mu e_{c\nu})[2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\gamma_5\gamma_d\, \epsilon^{dabc}] \tag{13}$$

By expanding the affine connection in terms of the Levi Civita connection $\widetilde{\nabla}$ and the contorsion tensor

$$\nabla_\mu e_{c\nu} = \widetilde{\nabla}_\mu e_{c\nu} - K^\lambda_{\nu\mu} e_{c\lambda}, \tag{14}$$

Equation (14) separates into a torsion free and a torsion dependent part the expression $\gamma^\mu\Omega_\mu$. By recalling the expression of the spin coefficients

$$\gamma_{bca} = e_b^\nu (\widetilde{\nabla}_\mu e_{c\nu}) e_a^\mu, \tag{15}$$

the resulting torsion free part of (13) can be reduced to the form:

$$\gamma^\mu \Big(\frac{1}{2}\gamma^{ad}_{\ \ a}\ e_{d\mu} + \frac{i}{4}\gamma_5 \epsilon^{dabc} e_{d\mu}\gamma_{bca}\Big) \tag{16}$$

Similarly, the resulting contortion dependent part of $\gamma^\mu\Omega_\mu$ in (13) is

$$-\frac{1}{4}\Big(\eta^{ab}\gamma^c - \eta^{ac}\gamma^b \quad -i\gamma_5\gamma_d\, \epsilon^{dabc}\Big) e_a^\mu\, e_b^\nu e_c^\lambda K_{\lambda\nu\mu} = \tag{17}$$

$$= -\frac{1}{4}\Big(g^{\mu\nu} K^{\ \lambda}_{\nu\mu}\gamma_\lambda + i\gamma_5\gamma_d e_a^\mu e_b^\nu e_c^\lambda K_{\lambda\nu\mu}\Big) \tag{18}$$

$$= -\gamma^\lambda \Big(\frac{1}{4}\tau_\lambda + \frac{3}{4} i\gamma_5 \mathcal{A}_\lambda\Big) \tag{19}$$

where γ_5 has been assumed to anti commute with the Dirac γ matrices. Finally the Dirac action can be written more explicitly

$$S_D = \int d^4x \sqrt{g}\ \overline{\psi}\Big\{i\gamma^\mu[\partial_\mu + i\gamma_5(A^G_{1\mu} + A^T_{1\mu}) + A^G_{2\mu} + A^T_{2\mu}] + m\Big\}\psi \tag{20}$$

where

$$A^G_{1\mu} = \frac{1}{4}\epsilon^{dabc}\gamma_{bca}e_{d\mu}, \qquad A^T_{1\mu} = -\frac{3}{4}\mathcal{A}_\mu \tag{21}$$

$$A^G_{2\mu} = \frac{1}{2}\gamma^{ad}_{\ \ a}\, e_{d\mu}, \qquad A^T_{2\mu} = -\frac{1}{4}\tau_\mu \tag{22}$$

The field equation of spin 1/2 particle coupled to gravity and torsion can now be obtained by varying the total action $S = S_D + S_{EHC}$ (with $S_D,\ S_{EHC}$ given by (20) and (10)), with respect to the fields $\overline{\psi}$, $U_{\alpha\mu\nu},\ \tau_\mu,\ \mathcal{A}^\mu$. One obtains a spin 1/2 field equation with constraints:

$$\gamma^\mu[\partial_\mu + i\gamma_5(A^G_{1\mu} + A^T_{1\mu}) + A^G_{2\mu} + A^T_{2\mu}]\psi = im\psi \tag{23}$$

$$U_{\alpha\mu\nu} = 0 \tag{24}$$

$$\tau^\mu = -\frac{3}{8}\, i\,\overline{\psi}\gamma^\mu\psi \tag{25}$$

$$\mathcal{A}^\mu = -\frac{1}{4}\,\overline{\psi}\gamma_5\gamma^\mu\psi \tag{26}$$

If one neglects the torsion terms $A^T_{1\mu}, A^T_{2\mu}$, the equation (23) can be interpreted as the Dirac equation in curved space time.

The form of equation (23) is not the most general spin 1/2 equation interacting with torsion and gravity (For this one can refer to e.,g. [2]). It seems however of interest that the non linear terms of the equation represent an interaction of the particle with its own current like terms (25), (26).

This is also clear in the following Sections where the results are translated into the language of the Newmann Penrose formalism.

4 Dirac equation in two spinor form

The tetrad e^μ_a of eq. (9) is now chosen so that

$$\eta_{ab} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0) \end{pmatrix} \tag{27}$$

One can then check that, by mimicking the procedure of the previous section, the same final results (20), (17) still hold, the only difference being that e^μ_a is now a null tetrad frame. To simplify notations it is convenient to denote $e^\mu_1 \equiv l^\mu$, $e^\mu_2 \equiv n^\mu$, $e^\mu_3 \equiv m^\mu$, $e^\mu_4 \equiv m^{\star\mu}$ (with l^μ, n^μ real; $m^{\star\mu} = m^{\mu\star}$). Therefore $l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = m^{\star\mu} m^\star_\mu = 0$, $l^\mu n_\mu = 1$, $m^\mu m^\star_\mu = -1$. Associated to the null tetrad frame there are then the 4×4 Dirac matrices

$$\gamma_\mu = \sqrt{2}\begin{pmatrix} 0 & G_\mu \\ G^+_\mu & 0) \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I_4 \tag{28}$$

(G^+ denotes here the adjoint of G, see e.g. [14] that still satisfy the usual anti commutation relations. This is a consequence of the definition of the G matrices in terms of the spin matrices

$$G^\mu \equiv \sigma^\mu_{AB'} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} l^\mu & m^\mu \\ m^{*\mu} & n^\mu) \end{pmatrix}, \tag{29}$$

The G_μ's are then Hermitian matrices that satisfy the relations $G_\mu G^\star_\nu + G_\nu G^\star_\mu = -g_{\mu\nu} I_2$. We further set $\psi \equiv [^{Q_{A'}}_{P_A}]$. As to the γ_5, it is the one of flat space-time whose representation is taken to be [14]: $\gamma_5 = \begin{pmatrix} -iI_2 & 0 \\ 0 & iI_2 \end{pmatrix}$ the two dimensional identity matrix [14]. To make explicit the Dirac equation (23) in the two spinor form one has to choose how act the σ, σ^+ matrices on the Dirac two spinor.

Case 1. A first choice gives:

$$\sqrt{2}\sigma^{\mu A}{}_{B'}\, \partial_\mu P_A - \sqrt{2}\, \sigma^{\mu A}{}_{B'} [A^G_{1\mu} + A^T_{1\mu} - A^G_{2\mu} - A^T_{2\mu}] P_A = im\, Q_{B'} \tag{30}$$

$$\sqrt{2}\, \sigma^{+\mu B'}_{A} Q_{B'} + i\sqrt{2}\, \sigma^{+\mu B'}_{A} (A^G_{1\mu} + A^T_{1\mu})(-i) Q_{B'} + \\ + \sqrt{2}\, \sigma^{+\mu B'}_{A} (A^G_{2\mu} + A^T_{2\mu}) Q_{B'} = im\, P_A \tag{31}$$

The expressions $\sqrt{2}\sigma^\mu_{BA'}\partial_\mu = \partial_{BA'}$ are the directional derivatives that, as usual, will be denoted by $\partial_{00'} = D = l^\mu \partial_\mu$, $\partial_{01'} = \delta = m^\mu \partial_\mu$, $\partial_{10'} = \delta^\star = m^{\star\mu}\partial_\mu$, $\partial_{11'} = \Delta = n^\mu \partial_\mu$.

Accordingly, the equation (30) (31) can be recast into the form

$$\partial_{AB'} P^A - \sqrt{2}\sigma^\mu_{AB'} P^A \Big[(A^G_{1\mu} - A^G_{2\mu}) + (A^T_{1\mu} - A^T_{2\mu}) P^A \Big] = -im Q_{B'} \tag{32}$$

$$\partial_{AB'} Q^{B'} - \sqrt{2}\sigma^\mu_{AB'} Q^{B'} \Big[(A^{G*}_{1\mu} - A^{G*}_{2\mu}) + (A^{T*}_{1\mu} - A^{T*}_{2\mu}) \Big] = -im P_A \tag{33}$$

Case 2. Another choice is to let both σ and σ^+ to act in the "same way" on the Dirac two-spinor. From (23), one then now obtains, instead of (32), (33), the two two spinor equations

$$\partial_{AB'} P^A - \sqrt{2}\sigma^\mu_{AB'} P^A \Big[(A^G_{1\mu} - A^G_{2\mu}) + (A^T_{1\mu} - A^T_{2\mu}) \Big] = -im\, Q_{B'} \tag{34}$$

$$\partial_{AB'} Q^A - \sqrt{2}\sigma^\mu_{AB'} Q^A \Big[(A^{G*}_{1\mu} - A^{G*}_{2\mu}) + (A^{T*}_{1\mu} - A^{T*}_{2\mu}) \Big] = im\, P_{B'} \tag{35}$$

The equations (32), (33) as well (34), (3), can be further expanded in terms of the spin coefficients. It is useful to recall that the spin coefficients are conventionally denoted by [11] (see also [4]):

$$\rho = \gamma_{314} \qquad \epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) \qquad \pi = \gamma_{241} \qquad \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) \tag{36}$$

$$\mu = \gamma_{243} \qquad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) \qquad \tau = \gamma_{312} \qquad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) \tag{37}$$

Then one can simplify the expression

$$\sqrt{2}\sigma^{\mu}_{AB'}(A^G_{1\mu} - A^G_{2\mu}) = \begin{pmatrix} \delta_{d2} & -\delta_{d4} \\ -\delta_{d3} & \delta_{d1} \end{pmatrix} (\frac{1}{4}\epsilon^{dabc}\gamma_{bca} - \frac{1}{2}\gamma^{ad}{}_{a}) \tag{38}$$

$$= \begin{pmatrix} \rho - \epsilon & \tau - \beta \\ \alpha - \pi & \gamma - \mu \end{pmatrix}_{AB'} \tag{39}$$

(δ_{di} the Kronecker delta) and the expression

$$\sqrt{2}\,\sigma^{\mu}_{AA'}\left(A^T_{1\mu} - A^T_{2\mu}\right) = \sqrt{2}\sigma^{\mu}_{AA'}\,(-\mathcal{A}_{\mu} + \frac{3}{4}\,\tau_{\mu}) \tag{40}$$

$$= \sqrt{2}\,\sigma^{\mu}_{AA'}\Big(\frac{3}{16}\overline{\psi}\gamma_5\gamma_{\mu}\psi - \frac{3}{32}i\,\overline{\psi}\gamma_{\mu}\psi\Big) \tag{41}$$

$$= -\frac{9}{16}\,i\,(\overline{Q}_A Q_{A'} + P_A\overline{P}_{A'}) \tag{42}$$

It should be noted that in the definition of $\overline{\psi} = \psi^{+}\gamma^{o}$ the matrix γ^0, is the constant special relativistic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and not the ”curved” one ([12]; see also [13].) In the last calculations it has also been used the relation $G_{AA'}{}^{a}G_{a}{}^{BB'} = \epsilon_A{}^B\epsilon_{A'}{}^{B'}$ [14].

It is now possible to expand the Dirac equation (32), (33) in terms of directional derivatives and spin coefficients.

4.1 Torsion free Dirac equation

By neglecting the torsion term (42), the two spinor Dirac equation can be made explicit in terms of directional derivatives and spin coefficients.

Case 1 The equations (32) (33) become, by use of (39),

$$(D + \epsilon - \rho)P_1 - (\delta^* + \pi - \alpha)P_0 = -imQ_0 \tag{43}$$

$$(\delta + \beta - \tau)P_1 - (\Delta + \mu - \gamma)P_0 = -imQ_1 \tag{44}$$

$$(D + \epsilon^* - \rho^*)Q_1 - (\delta + \pi^* - \alpha^*)Q_0 = -imP_0 \tag{45}$$

$$(\delta^* + \beta^* - \tau^*)Q_1 - (\Delta + \mu^* - \gamma^*)Q_0 = -imP_1 \tag{46}$$

The result is exactly the expanded form of the equations

$$\widetilde{\nabla}_{AB'}P^A = -i\,m\,Q_{B'}, \tag{47}$$

$$\widetilde{\nabla}_{AB'}Q^{B'} = -i\,m\,P_A \tag{48}$$

Note that by the substitution $m \to -im$ the equation (47), (48) coincide with the formulation of the Dirac equation as proposed in [14].

Case 2. By similar use of (40), from (34), (35), one obtains the expanded form of:

$$\widetilde{\widetilde{\nabla}}_{AB'}P^A = -i\,m\,Q_{B'}, \tag{49}$$

$$\widetilde{\widetilde{\nabla}}_{AB'}Q^A = i\,m\,P_{B'} \tag{50}$$

In such case, by the substitution $Q_B \to \bar{Q}_{B'}$ and by taking the complex conjugate of (50), one would obtain the formulation of the Dirac equation in two spinor form as proposed in Ref.[4] and that was also obtained in [17] by a different , but similar, procedure.

4.2 Dirac equation with torsion

If the torsion term (42) does not vanish, the complete expanded form of (32), (33) is then

$$\widetilde{\nabla}_{AB} P^A - icJ_{AB'} P^A = -i\, m\, Q_{B'}, \tag{51}$$
$$\widetilde{\nabla}_{AB'} Q^{B'} + icJ_{AB} Q^{B'} = -i\, m\, P_A, \quad c = 9/16 \tag{52}$$

where $J_{AB'} = \overline{Q}_A Q_{A'} + P_A \overline{P}_{A'}$ is the Dirac two spinor correspondent of the Dirac four current as it appears in (41-43). (Similarly in Case 2 one obtains the equations (50), (51) with the adjoint of current term as in (51), (52)).

The equations (51), (52) can be interpreted as the Dirac equation of a particle of mass m in a general curved space time with torsion where the action of the torsion (not the most general one indeed) results in an interaction of the particle with its own current. The current is still conserved $\widetilde{\nabla}_{AB'} J^{AB'} = 0$.

It is useful to notice that the constant "c" can be considered an arbitrary (real) constant to be specified according to physical application. To that end it suffices to consider a total action of the form $S = S_{EHC} + x\, S_D$, x a real number. Indeed, by the previous procedure, both $A^T_{1\mu}$ and $A^T_{2\mu}$, and hence "c", would result in a multiplication by "x" according to (40)-(42).

5 Remarks and comments

For what concerns the solution of the Dirac equation with torsion, it is evident that it is difficult to be obtained in general. In particular the equations (51), (52) remain coupled (contrarily to the torsion free case) also in the massless case. There are some particular situations that reduce the difficulty. For instance, in case of static metric it is easily seen that the time dependence of the solution factors out in a form like $\exp(ikt)$. This at least allows the separation of the equation also in presence of torsion (e., g. [23, 24]).

Particular or approximated solutions have been obtained in special case of physical interest. In flat space time the existence of standing wave solution was proved [21, 22]. The perturbation by torsion of the Hydrogen energy spectrum was seen to be irrelevant, at the present time, in standard cosmology [18]. As far as the author knows, the problem of the determination of the general solution of eqs. (51-52) is still open.

Finally, in the above scheme, one could consider also variation of the total action with respect to the gravity $g_{\mu\nu}$ so to have interaction among gravitational, Dirac, and torsion fields [19]. Solving the corresponding equations, it is a fortiori a difficult problem. However, in case of the Einstein - Dirac equation (absence of torsion) an elementary solution was given in the Robertson Walker space time [19]. Such solution is not of L^2 class so that there is not contradiction with the result of [20] for which the Einstein - Dirac equation has no quantum solutions in Robertson Walker space time.

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