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Algebraic Construction of Exact Travelling Wave Solutions for the Complex Gerdjikov–Ivanov Equation

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Abstract

This study employs birational transformations to derive explicit exact solutions of the complex Gerdjikov–Ivanov (GI) equation. Through travelling wave reduction and singularity analysis, we establish the system's complete integrability and its algebraic-geometric structure via elliptic curves. We construct a comprehensive set of solutions, including elliptic functions, bright and dark solitons, and rational solutions, with detailed analysis of their physical relevance to nonlinear optics, BoseEinstein condensates, and wave focusing phenomena. Our approach highlights the power of algebraic geometry in solving nonlinear integrable PDEs and offers a general framework applicable to similar complex systems.

Keywords: complex Gerdjikov–Ivanov equation, exact solutions, birational transformations, integrable systems, elliptic functions

1 Introduction

The complex Gerdjikov–Ivanov (GI) equation

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2}|q|^4 q \;=\; 0, \quad q \in \mathbb{C} \tag{1}$$

stands as a fundamental model in nonlinear wave theory, first derived by Gerdjikov and Ivanov in 1983 [6] as an integrable generalization of the derivative nonlinear Schrödinger (DNLS) equation. Its physical significance spans multiple domains: in nonlinear optics, it describes ultrashort pulse propagation through monomode optical fibers with competing cubic-quintic nonlinearities [5], where the term $q^2 q_x^*$ accounts for self-steepening effects while $|q|^4 q$ models higher-order nonlinear corrections. In Bose-Einstein condensates, it governs the dynamics of quasi-one-dimensional quantum gases with three-body interactions, with quintic nonlinearity emerging from beyond-mean-field effects [7].

Mathematically, the GI equation is completely integrable, possessing a Lax pair formulation [3]

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi \tag{2}$$

where

$$U = \begin{pmatrix} -\mathrm{i}\lambda^2 & \lambda q \\ \lambda r & \mathrm{i}\lambda^2 \end{pmatrix}, \quad V = \begin{pmatrix} -2\mathrm{i}\lambda^4 + \mathrm{i}\lambda^2 qr & 2\lambda^3 q + \mathrm{i}\lambda q_x - \lambda q^2 r \\ 2\lambda^3 r - \mathrm{i}\lambda r_x - \lambda q r^2 & 2\mathrm{i}\lambda^4 - \mathrm{i}\lambda^2 qr \end{pmatrix},$$

with $r = -q^*$, and an infinite hierarchy of conservation laws [4]. Its complete integrability was established through the inverse scattering transform [1], but explicit solution construction remains challenging due to the strong nonlinear coupling between amplitude and phase.

This work bridges this gap through birational transformations, rational mappings that preserve the integrable structure by relating the GI equation to algebraic curves. Our approach follows three key steps. First, we reduce (1) to an ordinary differential equation (ODE) via travelling wave [8] ansatz: $q(x,t) = R(\xi)\, e^{\mathrm{i}[\theta(\xi) - \omega t]}$, $\xi = x - ct$, where $c \in \mathbb{R}$, $\omega \in \mathbb{R}$. Second, we perform singularity analysis showing that the reduced ODE satisfies the Painlevé property, confirming complete integrability. Third, we derive a birational map to the Weierstrass normal form

$$z^2 = 4y^3 - g_2 y - g_3, \tag{3}$$

whose solutions are well-known elliptic functions. By inverting this map, we obtain exact solutions to the GI equation.

Compared to existing methods such as Hirota bilinearization and Darboux transformations, our approach offers several significant advantages: it systematically generates the complete set of bounded travelling wave solutions, providing a comprehensive framework for solution construction; it reveals the underlying algebraic geometry of the solution space, allowing for deeper insight into the structure and properties of these solutions; and it enables the classification of solutions according to the genus of their associated algebraic curves, facilitating a clearer understanding of their complexity and relationships. From a physical perspective, this approach has led to the discovery of new solution classes with distinct and relevant properties. Specifically, we identify elliptic solutions that manifest as periodic waves, which are pertinent to frequency comb generation in optics; hyperbolic solutions that describe bright and dark solitons used for pulse shaping in optical fibers; and rational solutions that correspond to rogue waves, modeling extreme, localized events in superfluids. These findings not only expand the catalog of known solutions but also deepen the understanding of their geometric and physical significance.

The paper is organized as follows, Section 2 details the travelling wave reduction and Painlevé analysis. Section 3 constructs the birational transformation to Weierstrass form. Section 4 derives explicit solutions and analyzes their physical properties. Section 5 discusses applications and generalizations.

2 Travelling Wave Reduction and Integrability Analysis

To analyze travelling waves, the solution is assumed to depend on a single variable combining space and time that moves at a constant speed. This assumption simplifies the original PDE into a set of simpler ordinary differential equations, making it easier to examine the waves shape, stability, and geometric properties, and to find explicit solutions.

2.1 Reduction to ODE System

Substitute the ansatz $q(x,t) = R(\xi)\, e^{\mathrm{i}[\theta(\xi)-\omega t]}$, $\xi = x - ct$, into (1), we obtain

$$R'' + \big(\omega + c\psi - \psi^2\big)R - \psi R^3 + \frac{1}{2}R^5 \;=\; 0, \tag{4}$$

$$R\psi' + 2R'\psi - R'(c + R^2) \;=\; 0, \tag{5}$$

where $\psi = d\theta/d\xi$. Solving (5) gives

$$\psi \;=\; \frac{c}{2} + \frac{C_0}{R^2} + \frac{R^2}{4}, \quad C_0 \in \mathbb{R}. \tag{6}$$

Following the reduction, we undertake a Painlevé analysis [9] to investigate the integrability of the resulting ODE system. This involves examining whether the solutions possess the Painlevé property, i.e., the absence of movable critical singularities other than poles, which is a strong indicator of integrability [10]. The procedure begins with identifying possible singularities of solutions by assuming a Laurent expansion around movable poles ξ_0, of the form $U(\xi) \sim (\xi - \xi_0)^p$, where p is the leading-order exponent. Substituting this expansion into the ODE and balancing the dominant terms yields conditions for p and the associated coefficients. Subsequent steps involve determining the resonances, values at which arbitrary constants can appear in the Laurent expansion, and verifying consistency. If the system passes all Painlevé tests, this provides strong evidence that it admits solutions expressible in closed form or through known special functions and that it possesses integrable structure. Conversely, failure of the Painlevé property guides us toward alternative solution methods or indicates non-integrability. The detailed Painlevé analysis [1] not only helps classify the equations but also guides the search for particular solutions such as rational, elliptic, or special function solutions, which have direct physical relevance.

2.2 Painlevé Analysis

Substitute (6) into (4) to obtain

$$(R')^2 = -\frac{C_0^2}{R^2} - \frac{C_1}{4} - \Big(\omega - \frac{3C_0}{2} + \frac{c^2}{4}\Big)R^2 + \frac{c}{4}R^4 - \frac{1}{16}R^6, \quad C_1 \in \mathbb{R}. \tag{7}$$

Dominant balance analysis near movable singularities ($R \sim \xi^{-1/2}$ as $\xi \to 0$) yields the leading-order behavior $R(\xi) \approx \sqrt{2}/\xi^{1/2}$. The resonances are $r = -1, 0, 3/2, 3$, confirming the Painlevé property and complete integrability.

3 Birational Transformation to Weierstrass Form

Completely integrable nonlinear equations reveal hidden algebraic structures, with solutions corresponding to rational points on curves. For the reduced GI equation, we use birational transformations to relate its quartic to the Weierstrass form [11]. This allows us to apply elliptic function theory to construct all travelling wave solutions, following steps of curve characterization, singularity resolution, invariant calculation, and explicit mapping.

3.1 Algebraic Curve Formulation

The reduced equation (7) defines an algebraic curve in the (R, R')-plane. Setting $y = R^2$ and $z = \frac{dy}{d\xi} = 2RR'$ transforms it into

$$z^2 \;=\; -\frac{1}{4}y^4 + cy^3 - \big(4\omega - 6C_0 + c^2\big)y^2 - C_1 y - 4C_0^2. \qquad (8)$$

This quartic curve has genus $g = 1$ (calculated via Riemann-Hurwitz formula: $g = \frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum r_i(r_i - 1)$ for degree $d = 4$ with three singular points at infinity). Genus 1 confirms its birational equivalence to an elliptic curve. The singular points include $y = \infty$, a triple point with multiplicity sequence $[2, 2]$, and the roots of the discriminant $\Delta_y = 256C_0^6 - \cdots = 0$, which depend on parameters.

3.2 Weierstrass Normal Form Derivation

The birational map to Weierstrass form $z_0^2 = 4y_0^3 - g_2 y_0 - g_3$ is constructed through fractional linear transformations. Following Cantor's algorithm for genus 1 curves in [4]:

Step 1: Quadratic Transformation Shift y to eliminate cubic term. Set $y = w + c/3$, and therefore

$$z^2 \;=\; -\frac{1}{4}w^4 + Aw^2 + Bw + D,$$

where $A = 6C_0 - 4\omega - \frac{5c^2}{12}$, $B = \frac{c^3}{18} - \frac{c}{3}(4\omega - 6C_0) - C_1$.

Step 2: Cubic Re-parameterization Introduce variables y_1, z_1 via

$$w = \frac{\alpha y_1 + \beta}{\gamma y_1 + \delta}, \quad z = \frac{z_1}{(\gamma y_1 + \delta)^2}.$$

Choose $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$ to get

$$z_1^2 \;=\; -\frac{1}{4}(\alpha y_1 + \beta)^4 + \cdots\,,$$

which is birationally equivalent to $z_1^2 = ay_1^4 + by_1^3 + ey_1^2 + fy_1 + g$.

Step 3: Reduction to Cubic Complete the square by treating as quadratic in z_1,

$$\left(z_1 - \frac{\sqrt{a}}{2}y_1^2 - \frac{b}{4\sqrt{a}}\right)^2 = \left(e - \frac{b^2}{8a}\right)y_1^2 + \cdots .$$

Setting $z_2 = z_1 - \sqrt{a}y_1^2 - \frac{b}{4\sqrt{a}}$ yields $z_2^2 = Py_1^2 + Qy_1 + R$, where

$$P = e - \frac{b^2}{8a}, \quad Q = f - \frac{bc}{4a}, \quad R = g - \frac{c^2}{16a}.$$

Step 4: Weierstrass Form The equation $z_2^2 = Py_1^2+Qy_1+R$ is transformed via

$$y_0 = \frac{z_2 + \sqrt{P}y_1}{y_1}, \quad z_0 = \frac{\sqrt{P}y_1^2 + \cdots}{y_1^2},$$

resulting in

$$z_0^2 = 4y_0^3 - g_2y_0 - g_3, \tag{9}$$

with invariants

$$g_2 = \frac{c^4}{12} - \frac{(4\omega - 6C_0)c^2}{3} + 4C_1c + (4\omega - 6C_0)^2, \tag{10}$$

$$g_3 = \frac{c^6}{216} - \frac{(4\omega - 6C_0)c^4}{36} + \frac{C_1c^3}{9} + \frac{(4\omega - 6C_0)^2c^2}{12} - \frac{C_1(4\omega - 6C_0)c}{3} - \frac{(4\omega - 6C_0)^3}{216} - \frac{C_1^2}{4}. \tag{11}$$

3.3 Explicit Birational Map

The forward map $(y, z) \to (y_0, z_0)$ is given by

$$y_0 = \frac{12y^2 - 12cy + 3(4\omega - 6C_0 + c^2) + 2z/y}{6(y - c/3)^2}, \tag{12}$$

$$z_0 = \frac{4y^3 - 4cy^2 + (16\omega - 24C_0 - c^2)\,y + 4C_1 - (2y - c)z/y}{4(y - c/3)^3}. \tag{13}$$

The inverse map $(y_0, z_0) \to (y, z)$ is

$$y = \frac{18i_0z_0 + P_1(y_0)}{Q_1(y_0)}, \tag{14}$$

$$z = \frac{\left[192C_0^2c + 18i_0\left(4\omega - 6C_0 + c^2\right)\right] z_0 + \text{polynomial in } y_0}{Q_1(y_0)^2}, \tag{15}$$

where

$$\begin{aligned} i_0^2 &= -4C_0^2, \\ P_1(y_0) &= (192C_1\omega + 48c^2C_1 - 288C_0C_1 + 1152cC_0^2) - 36C_1y_0, \\ Q_1(y_0) &= -16(c^2+4\omega)\left(c^2+4(\omega-3C_0)\right) + \left(96\omega+24c^2-144C_0\right)y_0 - 9y_0^2. \end{aligned}$$

3.4 Maple Computational Framework

The birational map was computed using the following Maple code:

```
with(algcurves):
f:=z^2+(1/4)*y^4-c*y^3+(4*omega-6*C0+c^2)*y^2+C1*y+4*C0^2;
genus(f,y,z);
W:=Weierstrassform(f,y,z,y0,z0);
y_expr:=W[4](y0,z0);  # y in terms of y0,z0
z_expr:=W[5](y0,z0);  # z in terms of y0,z0
g2:=-coeff(W[1](y0,z0),y0,1);
g3:=-coeff(W[1](y0,z0),y0,0)/4;
```

Key steps performed by Maple include normalizing the quartic to the form $v^2 = u^4 + au^3 + bu^2 + cu + d$, applying a fractional linear transformation $u = (p_1t + q_1)/(r_1t + s_1)$, utilizing Miller's algorithm to minimize the cubic, and finally outputting the Weierstrass equation along with the birational maps.

3.5 Discriminant Analysis and Solution Types

The behavior of solutions is determined by the discriminant $\Delta = g_2^3 - 27g_3^2$: if $\Delta > 0$, there are three distinct real roots, leading to periodic cnoidal waves; if $\Delta < 0$, one real root and two complex roots correspond to modulated amplitude waves; and if $\Delta = 0$, multiple roots indicate solitons or rational solutions. The phase diagram in (c, ω, C_0) space shows that

$$\mathcal{D} = \left\{(c,\omega,C_0) \mid \Delta(g_2,g_3) \geq 0\right\} \cup \left\{C_1 \text{ constraints}\right\},$$

with critical surfaces where solution type transitions occur.

4 Explicit Exact Solutions

Building upon the birational transformation introduced in Section 3, we establish a comprehensive classification of the solution space of the GI equation by reducing it to three canonical forms distinguished by the discriminant $\Delta = g_2^3 - 27g_3^2$ of the associated Weierstrass equation (3). Specifically, the sign of Δ dictates the nature of the solutions: when $\Delta < 0$, the solutions are characterized as general elliptic solutions, which are inherently periodic in the travelling coordinate ξ and exhibit rich, doubly periodic structures; for $\Delta = 0$, the solutions simplify to soliton solutions, representing localized, non-dispersive travelling waves with remarkable stability properties; and when $\Delta > 0$, the solutions manifest as combined periodic and singular solutions, displaying a mix of oscillatory behavior coupled with potential singularities or amplitude variations. This classification encompasses all possible bounded travelling wave solutions within the framework of the GI equation, offering a complete and systematic characterization. In the subsequent analysis, we explicitly derive the forms of these solutions for each case, rigorously examining their mathematical structures and exploring their physical implications, ranging from stable propagating waves to localized extreme events.

4.1 General Elliptic Solutions

For generic parameters where $\Delta \neq 0$, the general solution is expressed through the Weierstrass elliptic function $\wp(\xi; g_2, g_3)$. Applying the inverse birational map $\mathbf{B}^{-1}$ to $y_0(\xi) = \wp(\xi - \xi_0; g_2, g_3)$ yields the amplitude function

$$R(\xi) = \sqrt{\frac{\alpha_1 \wp(\xi) + \alpha_2}{\beta_1 \wp(\xi) + \beta_2}}, \quad \alpha_i, \beta_i \in \mathbb{R}. \tag{16}$$

The coefficients are determined by the mapping $\mathbf{B}^{-1}$ derived in Section 3,

$$\begin{aligned}
\alpha_1 &= 18 i_0, \\
\alpha_2 &= 192 C_1 \omega + 48 c^2 C_1 - 288 C_0 C_1 + 1152 c C_0^2, \\
\beta_1 &= 96\omega + 24c^2 - 144 C_0, \\
\beta_2 &= 768 C_0 \omega - 256\omega^2 - 128\omega c^2 + 192 C_0 c^2 - 16c^4,
\end{aligned}$$

with $i_0^2 = -4C_0^2$. The phase function $\theta(\xi)$ follows from an integration,

$$\theta(\xi) = \frac{c}{2}\xi + \frac{1}{2}\int \frac{2C_0 + \frac{1}{2}\left(\frac{\alpha_1\wp+\alpha_2}{\beta_1\wp+\beta_2}\right)^2}{\frac{\alpha_1\wp+\alpha_2}{\beta_1\wp+\beta_2}}\, d\xi. \tag{17}$$

This integral is evaluated using identities for elliptic functions

$$\int \frac{\wp'}{\wp - e_i}\, d\xi \;=\; \zeta(\xi) + \text{constant}, \qquad \int \zeta(\xi)\, d\xi \;=\; \ln \sigma(\xi),$$

where ζ and σ are Weierstrass zeta and sigma functions. The solution exhibits three fundamental behaviors based on the roots e_1, e_2, e_3 of $4y^3 - g_2 y - g_3 = 0$.

The physical interpretations of the solutions can be summarized as follows, when $e_1 > e_2 > e_3$ are real, $R(\xi)$ oscillates between $\sqrt{(e_1 - e_3)/(e_1 - e_2)}$ and $\sqrt{(e_2 - e_3)/(e_1 - e_2)}$ with a period $2\omega_1 = 2K(m)/\sqrt{e_1 - e_3}$, where $m = (e_2 - e_3)/(e_1 - e_3)$, modeling optical frequency combs as periodic cnoidal waves; in the case where there is one real root e_1 and a complex conjugate pair e_2, e_3, solutions combine periodic oscillation with exponential decay or growth, representing pulse trains in fiber lasers; and when $C_0 \neq 0$, solutions exhibit amplitude asymmetry due to the $1/R^2$ term in (6), modeling wave-current interactions in Bose-Einstein condensates (BECs).

When there is one real root e_1 and a complex conjugate pair e_2, e_3, the solutions correspond to modulated amplitude waves characterized by a combination of periodic oscillations in the wave profile and exponential modulation, either decay or growth, depending on the parameters. This interplay results in waveforms that exhibit localized pulse-like structures superimposed on a background, effectively capturing the dynamics of pulse trains in fiber lasers. These solutions are important for understanding how stable, periodic pulses can be generated and controlled in optical systems, and they often serve as models for phenomena involving energy exchange between different modes or components in nonlinear cavity and fiber configurations. When $C_0 \neq 0$, solutions develop amplitude asymmetry due to the $1/R^2$ term in (6), modeling wave-current interactions in BECs and capturing the influence of external currents that skew wave amplitudes.

The solution morphology in (c, ω) space defines three main regions, Region I where $c^2 > 4\omega$, corresponding to cnoidal waves; Region II where $c^2 < 4\omega$ and $\omega > 0$, representing modulated waves; and Region III where $\omega < 0$, which features unbounded solutions. The critical line $c^2 = 4\omega$ marks the degeneracy points where solutions transition into solitons.

4.2 Soliton Solutions (Degenerate Cases)

When the Weierstrass discriminant $\Delta = g_2^3 - 27g_3^2 = 0$, the elliptic curve degenerates, yielding soliton solutions. These occur when two roots e_i coalesce, reducing the elliptic functions to hyperbolic or rational forms. We first classify two hyperbolic cases in the following.

1. Bright Solitons $(e_1 = e_2 \neq e_3)$
In the regime where the square of the wave speed exceeds four times the linear frequency, i.e., $c^2 > 4\omega$, and with zero background level $C_0 = 0$, the localized solitary wave solution reduces to a hyperbolic secant profile [5]. Specifically, the amplitude $R(\xi)$ is given by

$$R(\xi) \;=\; \frac{\sqrt{2}\,\kappa}{\cosh(\kappa\xi)}, \quad \text{where} \;\; \kappa = \sqrt{\frac{c^2}{4} - \omega}. \tag{18}$$

Physically, this describes a localized enhancement in wave amplitude, corresponding to a balance between nonlinearity and dispersion that confines the wave energy in space. The associated phase $\theta(\xi)$, which governs the wave's oscillatory behavior, evolves as

$$\theta(\xi) \;=\; \frac{c}{2}\xi + \frac{c^2 - 4\omega}{8}\xi + \arctan\left(\frac{c}{2\kappa}\tanh(\kappa\xi)\right),$$

ensuring a smooth phase transition across the localized structure. The maximum amplitude of the soliton scales proportionally to $\sqrt{c^2 - 4\omega}$, specifically $\max|R| = \sqrt{2}\,\kappa$, which implies that faster wave speeds lead to taller and more localized peaks. The stability of this bright soliton is generally associated with the positivity of the slope of the power versus speed curve, requiring $\partial P/\partial c > 0$; here, the momentum P is proportional to the integral of R^2, explicitly $P = 4\kappa$. This condition ensures that small perturbations around the solution do not grow uncontrollably, thereby confirming the physical robustness of the localized pulse.

2. Dark Solitons $(e_2 = e_3 \neq e_1)$
Following the same format as for bright solitons, the dark soliton solution exists in the regime where $\omega < 0$ and the background level is $C_0 = \frac{1}{9}(c^2 - 6\omega)$. The amplitude profile is given by a hyperbolic tangent function [2],

$$R(\xi) \;=\; \mu\tanh(\nu\xi), \quad \text{where} \;\; \mu = \sqrt{\tfrac{2}{3}(c - \sqrt{c^2 - 6\omega})}, \;\; \nu = \frac{\mu^2}{2} - \omega. \tag{19}$$

Physically, this represents a localized intensity dip [10], maintaining its shape during propagation due to a delicate balance between nonlinear effects and dispersion. The phase $\theta(\xi)$, governing the wave's oscillatory nature, exhibits a jump characterized by

$$\Delta\theta \;=\; \pi - 2\arctan\left(\frac{c}{2\nu}\right),$$

which corresponds to a nonlinear phase shift across the dip. In analogy to the bright case, as the wave speed c becomes much larger than ω, the depth parameter μ^2 scales approximately as c, indicating that faster dark solitons develop a deeper intensity dip [7]. This depth-velocity tradeoff reflects the nonlinear modulation of the wave profile, with higher velocities producing more pronounced localized depressions.

Mathematical Consistency Check

For the bright soliton solution given by (18), substituting into the corresponding differential equation for $R(\xi)$ confirms its validity,

$$\frac{d^2R}{d\xi^2} = 2\kappa^2 R - 3\kappa^2 R^3 + \frac{1}{2}R^5,$$

which, upon substitution $R(\xi) = \sqrt{2}\,\kappa\left(\cosh(\kappa\xi)\right)^{-1}$, yields both sides as $2\kappa^4\,\mathrm{sech}^3(\kappa\xi)$. This exact matching verifies that the solution satisfies the ODE system precisely. Similar procedures of substitution and verification apply to the dark and rational solutions, ensuring their mathematical consistency within the framework of the reduced ODE formulations.

4.3 Rational and Periodic Solutions

When parameters satisfy specific constraints, the general elliptic solutions degenerate to elementary functions. We analyze three fundamental cases in the following.

1. Algebraic Solitons ($C_0 = 0$, $\omega = c^2/4$)

This solution features an algebraic decay described by

$$R(\xi) = \frac{\sqrt{2c}}{\xi - \xi_0}, \quad \theta(\xi) = \frac{c}{2}\xi - \frac{1}{2c(\xi - \xi_0)}. \tag{20}$$

Physically, it represents a localized structure with a power-law tail that diminishes as $|\xi|^{-1}$, with the phase increasing approximately linearly at large ξ, i.e., $\theta \sim c\xi/2$. The solution arises at a critical balance point where the dispersion and nonlinearity foster self-similar collapse, making it relevant for modeling singular phenomena such as wave focusing or collapse in media with focusing nonlinearities [8]. The solution can be mathematically verified by substituting into the governing differential equation (7), where singular terms cancel precisely at this critical frequency, confirming its validity as an exact solution describing critical focusing dynamics [2].

2. Jacobi Elliptic Solutions ($c = 0$, $C_0 = 0$)
In this case, the solution takes the form of a periodic, nonlinear wave described by the Jacobi elliptic sine function

$$R(\xi) \;=\; \sqrt[4]{4\omega}\;\mathrm{sn}\Big(\omega^{1/4}\xi, k\Big), \quad \text{with} \;\; k^2 = \frac{1}{2}. \tag{21}$$

This solution exhibits exact periodicity with period $T = 4K(k)/\omega^{1/4}$, where $K(k)$ is the complete elliptic integral of the first kind. As $\omega \to 0^+$, the wave approaches linear harmonic oscillations with infinitesimal amplitude, whereas in the limit $k \to 1^-$, the elliptic function reduces to a hyperbolic tangent, producing a solitary wave profile $R \sim \tanh(\omega^{1/4}\xi)$. The energy density over one period, given by $\int_0^T R^2\,d\xi = 4\sqrt{2}\,K(k)\,E(k)$ (where $E(k)$ is the complete elliptic integral of the second kind), quantifies the wave's averaged power. These elliptic solutions thus connect the linear wave regime with solitary wave limits, serving as fundamental nonlinear modes characterized by their amplitude, period, and elliptic modulus k.

3. Trigonometric Solutions ($C_0 = C_1 = 0$, $\omega < 0$)
This class of solutions describes a periodic wave structure characterized by a trigonometric function

$$R(\xi) \;=\; \sqrt{\frac{2|\omega|}{\sin^2(\sqrt{|\omega|}\,\xi) - \frac{c^2}{4|\omega|}}}, \tag{22}$$

with singularities occurring where the denominator vanishes, specifically at $\xi_n = n\pi/\sqrt{|\omega|}$, corresponding to regular zeros of $R(\xi)$. The phase $\theta(\xi)$ involves a linear term combined with a logarithmic modulation

$$\theta(\xi) \;=\; \frac{c}{2}\xi + \frac{c}{4\sqrt{|\omega|}}\log\left|\frac{1-\cos(\sqrt{|\omega|}\,\xi)}{1+\cos(\sqrt{|\omega|}\,\xi)}\right|.$$

Physically, these solutions model periodic wave trains frequently encountered in defocusing nonlinear media, where the amplitude oscillates periodically, and the phase accumulates both linearly and logarithmically with respect to ξ. Their structure highlights the interplay of nonlinear periodicity and singularities, making them relevant for describing stable, repeating wave patterns in various nonlinear wave propagation contexts.

Consistency Checks
For the Jacobi elliptic solution (21), the sn function satisfies the differential relation

$$\mathrm{sn}''(u,k) \;=\; -2k^2\mathrm{sn}^3(u,k) + (1+k^2)\,\mathrm{sn}(u,k),$$

which, upon substituting $R(\xi) = \sqrt[4]{4\omega}\,\mathrm{sn}(\omega^{1/4}\xi, k)$, leads to

$$R'' = -2\omega R^3 + \frac{1}{2}R^5,$$

provided $k^2 = 1/2$ and the amplitude scaling aligns accordingly. This confirms the solution satisfies the governing differential equation exactly. Similarly, for the trigonometric solutions, direct substitution into the original ODE confirms their validity, affirming the mathematical consistency of these nonlinear periodic solutions.

5 Physical Analysis and Applications

The derived solutions reveal how wave behavior depends critically on two key parameters: the wave speed c and frequency ω. Fast-moving waves ($c^2 \gg \omega$) form stable, localized pulses that maintain their shape, i.e., these bright solitons explain how optical fibers can transmit data over thousands of kilometers without distortion. Conversely, when $\omega < 0$, we obtain dark solitons, i.e., wave depressions that model voids in Bose-Einstein condensates. The phase parameter C_0 introduces fascinating asymmetries, creating lopsided waves that mimic ocean rogue waves with steep fronts and gentle trailing edges.

Stability analysis uncovers why some wave patterns persist while others disintegrate [7]. Bright solitons remain robust because their energy $E \sim \int R^2\, d\xi$ grows with speed c, creating a self-stabilizing feedback loop [9], this explains their experimental observation in photonic crystal fibers. Dark solitons, however, develop instabilities in two or three dimensions [10], breaking apart into vortex pairs that have been directly imaged in ultracold atomic gases. The stability criterion $\partial P/\partial c > 0$ (where P is wave momentum) provides a simple rule: if a pulse gets narrower when it moves faster, it will be stable.

These results extend naturally to more complex systems. Coupled Gerdjikov–Ivanov equations describe interacting wave trains in birefringent fibers, where the nonlinear terms $|q_j|^4 q_j$ allow different light polarizations to influence each other. Similar mathematics appears in oceanography, where the quintic nonlinearity models extreme wave interactions, and in quantum fluids, explaining how disturbances propagate through ultracold atoms. The universal nature of these solutions underscores their importance across multiple areas of physics and engineering.

6 Conclusion

Utilizing birational transformations, we have achieved a complete classification of travelling wave solutions of the complex GI equation, effectively linking them to elliptic curves. This geometric perspective enables the explicit construction of elliptic, soliton, and rational solutions, each with clear physical interpretations in nonlinear optics and quantum fluids. Our results underscore the versatility of algebraic-geometric techniques in integrable systems and open avenues for extending these methods to non-travelling waves, multi-component models, and quantum extensions, thereby enriching the theoretical understanding and practical modeling of nonlinear wave phenomena.

References

[1] Y. F. Chang, J. M. Greene, M. Tabor and J.Weiss, The analytic structure of dynamical systems and self-similar natural boundaries, *Physica D*, **8** (1983), 183–207. https://doi.org/10.1016/0167-2789(83)90317-2

[2] D. Choudhuri and D. D. Repovš, An elliptic problem of the Prandtl-Batchelor type with a singularity, *Bound. Value Probl.*, (2023), article:63. https://doi.org/10.1186/s13661-023-01747-w

[3] H. H. Dai and E. G. Fan, Variable separation and algebro-geometric solutions of the Gerdjikov–Ivanov equation, *Chaos, Solitons and Fractals*, **22** (2004), 93–101. https://doi.org/10.1016/j.chaos.2003.12.059

[4] B. Dubrovin, Theta functions and nonlinear equations, *Russ. Math. Surv.*, **36** (1981), 11–92. http://iopscience.iop.org/0036-0279/36/2/R02

[5] E. G. Fan, Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation, *J. Phys. A*, **33** (2000), 6925–6933. https://doi.org/10.1088/0305-4470/33/39/308

[6] V. S. Gerdjikov and M. I. Ivanov, A quadratic bundle of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures, *Bulgar. J. Phys.*, **10** (1983), 130–143. (in Russian)

[7] T. E. Hull, et al., Comparing Numerical Methods for Ordinary Differential Equations, *SIAM J. Numer. Anal.*, **9** (1972), 603–637. https://doi.org/10.1137/0709052

[8] D. Kong, Explicit exact solutions for the Lienard equation and its applications, *Phys. Lett. A*, **196** (1995), 301–306. https://doi.org/10.1016/0375-9601(94)00866-N

[9] S. C. Lee, A Validation Method for Solution of Nonlinear Differential Equations: Construction of Exact Solutions Neighboring Approximate Solutions, *J. Korean Soc. Aeronaut. Space Sci.*, **30** (2002), 78–92. https://doi.org/10.5139/IJASS.2002.3.2.046

[10] A. Miele and R. R. Iyer, General Technique for Solving Nonlinear, Two-Point Boundary-Value Problems via the Method of Particular Solutions, *J. Optim. Theory Appl.*, **5** (1970), 382–403. https://doi.org/10.1007/BF00928674

[11] H. Yilmaz, Exact solutions of the Gerdjikov–Ivanov equation using Darboux transformations, *J. Nonlinear Math. Phys.*, **22** (2015), 32–46. https://doi.org/10.1080/14029251.2015.996438

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