Advanced Studies in Theoretical Physics
Vol. 18, 2024, no. 1, 31 - 40
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/astp.2024.92089

Scalar Field Equation via the Newman Penrose Formalism and Separation in the Kerr and Lemaître-Tolman-Bondi Metric

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Abstract

The scalar field equation is expanded by the Newman Penrose formalism in a general curved space time. The result is specialized to the Kerr and the Lemaître-Tolman-Bondi metric. The equation results finally in known expression. It is directly separable in Kerr metric and in a class of Lemaître-Tolman-Bondi cosmological model. The problem of the determination of the normal mode solutions is possible within the Lemaître-Tolman-Bondi cosmological model.

Keywords: Kerr and LTB metric; Newman Penrose formalism; Scalar Field Equation; Separation

1 Introduction

The scalar field equation is a spin zero relativistic equation. It has been widely studied not only because it is the lowest spin value equation, but also for applications that range from quantum mechanics to cosmology.

From a mathematical point of view, the equation is generally developed in the coordinate formalism in terms of partial derivatives. To get the final form of the equation one could as well proceed by the Newman-Penrose formalism [1, 2], that is a special case of the tetrad formalism. Accordingly the expansion of covariant spinor derivatives involves the directional derivatives and the spin coefficients [1]. This is a cumbersome method when applied to the scalar field equation, but it seems very useful in the formulation and separation of higher spin value equations [3, 4, 6, 5]. Since explicit null tetrad frames are well known in many cases of physical space time metric (e.g. [7]), it seems useful to study the scalar field equation by the Newman Penrose formalism. Accordingly, the scalar field equation is first expanded in a general curved space time. The general result is then made explicit in Kerr and in Lemaître-Tolman-Bondi space time. These are metrics of physical interest not only in themselves, but also because they contain the Schwarzschild and the Robertson-Walker metric respectively. In all cases the final result coincides with that directly obtained by the pure coordinate formalism. Separability of the equation directly follows in the Kerr metric. In LTB metric, separability holds for a subclass of LTB cosmologies. Some comments on the determination of the normal modes of the equation are also done that are of relevance in view of the quantizaton of the scalar field.

2 Basic formalism

The scalar field equation is considered here in the general 4-dimensional Riemann space time. The study is based on the torsion free Newman Penrose 2-dimensional spinor formalism. It is based on the choice of a null tetrad frame of suitable normalized contravariant four vectors $\{l^i, n^i, m^i, \bar{m}^i\}$ [7]:

$$l^i\, m_i \;=\; l^i\, \bar{m}_i \;=\; n^i\, m_i \;=\; n^i\, \bar{m}_i = 0 \tag{1}$$

$$l^i l_i \;=\; n^i n_i \;=\; m^i m_i \;=\; \bar{m}^i \bar{m}_i = 0 \tag{2}$$

$$l^i n_i \;=\; 1, \quad m^i \bar{m}_i \;=\; -1 \tag{3}$$

The directional derivatives are customary denoted:

$$D = l^i \partial_i = \partial_{00'}, \qquad \Delta = n^i \partial_i = \partial_{11'} \tag{4}$$

$$\delta = m^i \partial_i = \partial_{01'}, \qquad \delta^* = \bar{m}^i \partial_i = \partial_{10'} \tag{5}$$

The following relations are useful:

$$Y_a \;=\; \sigma_a^{AB'} Y_{AB'}; \qquad \nabla_{AB'}\phi = \partial_{AB'}\phi \tag{6}$$

$$\partial_{AB'}\partial^{AB'} \;=\; D\Delta + \Delta D - \delta\delta^* - \delta^*\delta \tag{7}$$

where $\phi = \phi(x)$ is a scalar function, Y_a a four dimensiona vector and the $\sigma_a^{AB'}$'s the van der Waerden matrices [2].

3 Scalar field in Newman Penrose formalism

In space time of metric tensor $g_{\mu\nu}$ the Lagrangian of the (minimally coupled) scalar field $\phi(x)$ of mass μ, the Lagrangian equation and the torsion free scalar field equation reads (e. g., [8]):

$$L = \sqrt{|g|}\{\frac{1}{2}\nabla^{AX'}\nabla_{AX'}\psi - \frac{1}{2}\mu^2\psi^2\} \tag{8}$$

$$\frac{\partial L}{\partial \psi} - \nabla_{AX'}\frac{\partial\, L}{\partial(\nabla_{AX'}\psi)} = 0 \tag{9}$$

$$\nabla_{BY'}\nabla^{BY'}\psi + \mu^2\psi = 0 \tag{10}$$

The covariant spinor derivatives can be made explicit in terms of directional derivatives and spin coefficients. One has

$$\nabla_{AX'}\nabla^{AX'}\psi = \sigma^a_{AX'}\nabla_a(\partial^{AX'}\psi) \tag{11}$$

$$= \partial_{AB'}\partial^{AB'}\psi + \Gamma_{DX'B}^{\ \ D}(\partial^{BX'}\psi) + (\Gamma_{X'AX'_0}^{\ \ X'})^*(\partial^{AX'_0}\psi) \tag{12}$$

By using the conventional notation for the spin coefficient Γ, (see, e.g., the table 4.5.16 in [2]) one has then

$$\nabla_{AX'}\nabla^{AX'}\psi = \partial_{AB'}\partial^{AB'}\psi + (\epsilon - \rho + \epsilon^* - \rho^*)\partial^{00'}\psi$$
$$+(\mu - \gamma + \mu^* - \gamma^*)\partial^{11'}\psi + (\beta - \tau + \pi^* - \alpha^*)\partial^{01'}\psi$$
$$+(\pi - \alpha + \beta^* - \tau^*)\partial^{10'}\psi \tag{13}$$

The general form of the scalar field equation in the Newman Penrose formalism in curved space time is therefore

$$(D\Delta + \Delta D - \delta\delta^* - \delta^*\delta)\psi + \mu^2\psi + X\psi = 0 \tag{14}$$

$$X = (\epsilon - \rho + \epsilon^* - \rho^*)\Delta + (\mu - \gamma + \mu^* - \gamma^*)D$$
$$-(\beta - \tau + \pi^* - \alpha^*)\delta^* - (\pi - \alpha + \beta^* - \tau^*)\delta \tag{15}$$

The general equation is now made explicit in space time of physical interest.

4 Scalar field in Kerr space time

In the present case we refer to Chandrasekhar's book for notation and preliminary results. Accordingly the assumed metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{bmatrix} 1 - 2Mr/\rho^2 & 0 & 0 & 2aMr\sin^2\theta/\rho^2 \\ 0 & -\frac{\rho^2}{\triangle} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 2aMr\sin^2\theta/\rho^2 & 0 & 0 & -\sin^2\theta[r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\rho^2}] \end{bmatrix} \tag{16}$$

where $\triangle = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$. The null tetrad frame assumed here is the one considered in [7]. The corresponding directional derivatives, that are reported for the reader convenience, are:

$$D = \frac{r^2+a^2}{\triangle}\partial_t + \partial_r + \frac{a}{\triangle}\partial_\varphi, \qquad \Delta = \frac{r^2+a^2}{2\rho^2}\partial_t - \frac{\triangle}{2\rho^2}\partial_r + \frac{a}{\rho^2}\partial_\varphi, \quad (17)$$

$$\delta = \frac{ia\sin\theta}{\bar\rho\sqrt{2}}\partial_t + \frac{i}{\bar\rho\sqrt{2}}\partial_r + \frac{ia}{\bar\rho\sqrt{2}\sin\theta}\partial_\varphi, \qquad \delta^* = (\delta)^* \quad (18)$$

where $\bar\delta = r + ia\cos\theta$. The explicit expression of the spin coefficients, can be found in [7]. (Notice that the conventional rotation coefficient ρ is denoted by $\tilde\rho$ in the present Kerr metric case).

We now show that the scalar field equation is completely separable by variable separation. Since the coefficients of the metric tensor are t,φ independent, so are the coefficients of the scalar field equation. Therefore, by setting

$$\psi(t,r,\theta,\varphi) = \phi(r,\theta)\ e^{ikt+im\varphi}, \quad k,m \in \mathbf{R} \quad (19)$$

we first consider the effect of the different terms of (14) on ψ. By denoting $F_x(x,y) \equiv \partial_x F(x,y)$ one obtains :

$$(D\Delta + \Delta D)\phi = \Big\{\frac{\alpha}{2\rho^2}(A\phi + \phi_r) + \frac{\alpha}{\triangle}(B\phi - \frac{\triangle\phi_r}{2\rho^2})$$
$$- \frac{\triangle}{2\rho^2}\Big[(A\phi)_r + \phi_{rr}\Big] + (B\phi)_r - \partial\Big(\frac{\triangle}{2\rho^2}\phi_r\Big)\Big\} \quad (20)$$

$$(\delta\delta^* + \delta^*\delta)\psi = \frac{1}{\bar\rho\bar\rho^*}\Big\{\phi_{\theta\theta} - C^2\phi - \frac{ia\sin\theta}{\bar\rho\bar\rho^*}(rC\phi - ia\,cos\theta\,\phi_\theta)\Big\} \quad (21)$$

$$X\phi = \left\{\phi\Big[A\frac{M-r}{\rho^2} + \frac{2rB}{\rho^2}\Big] - \frac{ia\sin\theta\, ia\cos\theta}{\rho^4} - \frac{iarC\sin\theta}{\rho^4} - \frac{\cot\theta}{\rho^2}\phi_\theta\right\} \quad (22)$$

where it has been set:

$$A = \frac{\alpha}{\triangle}, \quad B = \frac{\alpha}{2\rho^2}, \quad \alpha = ik(r^2+a^2) + iam \quad (23)$$

$$C = ak\sin\theta + \frac{m}{\sin\theta} = C(\theta) \quad (24)$$

Reconstructing the equation (14), many terms simplify so that one is left with

$$-\phi_{\theta\theta} - \cot\theta\,\phi_\theta - \partial_r(\triangle\partial_r\phi) + (2AB + \mu^2)\rho^2\phi = 0 \quad (25)$$

By the explicit expression of A, B, ρ^2 the last equation separates by setting $\phi(r,\theta) = R(r)\,S(\theta)$ with separation constant λ_m to obtain:

$$S" + \cot\theta\, S' + \big[a^2(k^2-\mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda_m\big]S = 0 \quad (26)$$

$$\triangle\partial_r\big(\triangle\partial_r R\big) + \big[k^2(a^2+r^2)^2 + 4maMkr - (\mu^2 r^2 + \lambda_m + a^2k^2)\triangle + a^2m^2\big]R = 0 \quad (27)$$

The result coincides with those obtained in [9] and similarly in [10, 11] in the coordinate formalism. The equations (26), (27) have been discussed in connection with the quantization of the scalar field in Kerr metric [9, 11]. For recent studies concerning the solution see e.g., [12, 13] and recently [14].

The previous scheme contains the Schwarzschild case by choosing $a = 0$ in (16), (26), (27). The solution of the field equation is then of the form $\phi(t, r, \theta, \varphi) = \exp(ikr)Y_{lm}(\theta, \varphi)R(r)$ where Y_{lm}'s are the spherical harmonics and $R(r)$ satisfies the equation:

$$\frac{d^2R}{dr^2} + \frac{2(r-M)}{r(r-2M)}\frac{dR}{dr} + \Big[\frac{k^2r^2}{(r-2M)^2} - \frac{\lambda + m^2r^2}{r(r-2M)}\Big]R = 0 \tag{28}$$

($\lambda = l(l+1)$). An integration by series of the last equation has been given in [15] were also a discussion of the solutions has been done to characterize the existence of a set of normal mode of the complete scalar field.

The complete solution of the Schwarzschild case has been given in [16].

5 Lemaître-Tolman-Bondi space time

The LTB metric tensor $g_{\mu\nu}$ is given by (see, e. g.,[17]):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^\Gamma dr^2 - Y^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{29}$$

with $\Gamma = \Gamma(r,t)$, $Y = Y(r,t)$. The directional derivatives assumed here are those relative to the null tetrad frame given in [18]:

$$D = \frac{1}{\sqrt{2}}\Big(\partial_t + e^{\Gamma/2}\,\partial_r\Big), \qquad \Delta = \frac{1}{\sqrt{2}}\Big(\partial_t - e^{\Gamma/2}\,\partial_r\Big), \tag{30}$$

$$\delta = \frac{1}{Y\sqrt{2}}\Big(\partial_\theta + i\csc\theta\,\partial_\varphi\Big), \quad \delta^* = \frac{1}{Y\sqrt{2}}\Big(\partial_\theta - i\csc\theta\,\partial_\varphi\Big) \tag{31}$$

with corresponding non zero spin coefficients

$$\mu = \frac{1}{Y\sqrt{2}}\Big(Y_t - Y_r\,e^{-\Gamma/2}\Big), \quad \rho = -\frac{1}{Y\sqrt{2}}\Big(Y_t + Y_r\,e^{-\Gamma/2}\Big) \tag{32}$$

$$\beta = -\alpha = -\frac{\cot\theta}{2Y\sqrt{2}}, \qquad \epsilon = -\gamma = -\frac{\Gamma_t}{4\sqrt{2}} \tag{33}$$

In the present case one has ($\psi = \psi(t, r, \theta, \varphi)$):

$$(D\Delta + \Delta D)\psi = \phi_{tt} - e^{-\Gamma}\psi_{rr} - \frac{\Gamma_r}{2}e^{-\Gamma}\psi_r \tag{34}$$

$$(\delta\delta^* + \delta^*\delta)\psi = \Big(\partial_{\theta\theta} + \frac{1}{\sin^2\theta}\partial_{\varphi\varphi}\Big)\psi \tag{35}$$

$$X\psi = \Big(\frac{\Gamma_t}{2} + 2\frac{Y_t}{Y}\Big)\psi_t - \frac{\cot\theta}{Y^2}\psi_\theta - 2\,e^{-\Gamma}\frac{Y_r}{Y}\psi_r \tag{36}$$

The final form of the equation in the partial derivatives is then

$$\psi_{tt} - e^{-\Gamma}\psi_{rr} - \frac{1}{Y^2}\Big[\psi_{\theta\theta} + \frac{1}{\sin^2\theta}\psi_{\varphi\varphi} + \cot\theta\,\psi_\theta\Big] + \psi_t\Big[\frac{\Gamma_t}{2} + 2\frac{Y_t}{Y}\Big] + e^{-\Gamma}\psi_r\Big[\frac{\Gamma_r}{2} - 2\frac{Y_r}{Y}\Big] + \mu^2\psi = 0 \tag{37}$$

that separates by setting $\psi = \chi(\theta,\varphi)\phi(r,t)$. The angular equation gives $\chi(\theta,\varphi) = Y_{lm}(\theta,\varphi)$, the spherica harmonics, so that one is left with the r,t equation:

$$\phi_{tt} - e^{-\Gamma}\phi_{rr} + e^{-\Gamma}(\frac{\Gamma'}{2} - 2\frac{Y'}{Y})\phi_r + (\frac{\dot{\Gamma}}{2} + 2\frac{\dot{Y}}{Y})\phi_t + (\frac{\lambda}{Y^2} + \mu^2\phi) = 0 \tag{38}$$

that, coincides with what obtained in [19].

Further consideration relative to the separability of scalar field equation (38) can be developed within the Lemaître-Tolman-Bondi cosmological models.

Such model represents the general spherically symmetric solution of the Einstein field equation for a universe filled with freely falling dust like matter [17] here expressed by the metric tensor (29). By first integration of the Einstein field equation, the LTB cosmological model can be equivalently expressed by the equations [17]:

$$e^\Gamma = \frac{Y'^2}{1+2E(r)}, \quad \frac{\dot{Y}^2}{2} - \frac{M(r)}{Y} = E(r) \tag{39}$$

$$M(r) = 4\pi G\int_0^r dr\, Y^2\, Y' d(t,r) \tag{40}$$

$E(r), M(r)$ are arbitrary integration functions , $d(t,r)$ is the energy density of the dust matter. The integration of the Newton-like equation is given in parametric form by (e,.g.[17]):

$$\begin{aligned} Y &= G\frac{M(r)}{2E(r)}(\cosh\eta - 1) & (\eta > 0) \\ t - t_0(r) &= G\frac{M(r)}{[2E(r)]^{3/2}}(\sinh\eta - \eta) & (E > 0) \end{aligned} \tag{41}$$

$$\begin{aligned} Y &= G\frac{M(r)}{[-2E(r)]}(1 - \cos\eta) & (0 \le \eta \le 2\pi) \\ t - t_0(r) &= G\frac{M(r)}{[-2E(r)]^{3/2}}(\eta - \sin\eta) & (E < 0) \end{aligned} \tag{42}$$

$$Y = \Big[\frac{3}{2}(2M(r))^{1/2}(t - t_0(r))\Big]^{2/3} \qquad (E = 0) \tag{43}$$

where $t_0(r)$ is an arbitrary integration function.

Coming back to the eq. (38), by using the parametric representations (41-43) of the solution of the Newton like cosmological equation, the equation (38) results separable for $t_0(r) = 0$ both in case $E(r) = 0$, $M(r) \neq 0$ and in case $E \neq 0, |E|^{3/2} \propto M$ (see e., g. [19]). [Notice that, in both cases, the "physical radius" Y has a factorized dependence of the r, t variables as it is evident by an inspection into eqs. (41)-(43)].

Moreover, in case $E \neq 0$ the final separated radial equation can be recast into the form of an eigenvalue problem of a Weyl-Stone operator that results to be essentially self adjoint for a (non void) class of $E(r)$ functions. Accordingly, for such $E(r)$, there exists a set of normal modes of the complete scalar field equation [19]. The result is useful in view of the quantization of the scalar field in Lemaître-Tolman-Bondi cosmological models [20].

A second remark concerns the solution of the scalar field equation in the Robertson Walker space time. It can be obtained by choosing, in the previous scheme, $\exp\Gamma = R^2(t)/(1-\kappa r^2), Y = rR(t),\ \kappa = 0, \pm 1$, where $R(t)$ is the radius of the Universe. The resulting scalar field equation can be completely separated by elementary variable separation. The separated angular and radial equations can be explicitly integrated. Instead the separated time equation can be integrated only by giving an explicit cosmological time evolution. In general, a complete set of normal modes of the complete scalar field can be obtained. This follows by using a scalar product of the solutions, induced by a conserved current. Such results are well known and were already summarized in [21].

6 Comments

In the previous Sections it has been shown that the scalar field equation formulated by the Newman Penrose formalism can be made explicit, so to coincide to the usual one in coordinate variables, in some physical metric cases.

It is known that the Newman Penrose formalism is a powerful tool to study field equations of arbitrary spin in curved space time. In particular, for higher spin value, it has been put in evidence that there is a recurrence relation among the component equations of the field. This enables to reduce the solution of the massive field equation to a set of separated ordinary differential equations, thus making possibly easier to solve the field equation. In this line the massive spin 1/2 and the massless spin 1 field equation were widely studied and solved in Kerr metric [7]. Also the arbitrary spin massless field equation can be integrated in Kerr metric and more generally in Petrov type D space time [22, 23]. The arbitrary spin massive field equation were separated in Schwarzschild metric [4], in a class of LTB cosmologies [5] and in the Robertson Walker metric in which case the equation has also been integrated in term of Heun functions [6].

What lacks, for what concerns the previous considerations, is the treatment of spin 0 field equation in curved space time by the Newman Penrose formalism. Therefore, besides the mathematical interest, the present paper is intended to cover that lack that, so far, does not seem to have been done in the literature.

Acknowledgments. The author is grateful for the kind hospitality at the Department of Physics, Milan University. He took also advantage of the use of the computer resources of the INFN, Milan Section.

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Received: January 17, 2024; Published: January 31, 2024