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# Quantization of Lemaître-Tolman-Bondi Cosmology Based on the Associated Kepler-like Equation

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**Abstract**

The quantization of Lemaître Tolman Bondi Cosmology is reconsidered. The associated one dimensional Kepler like cosmological equation, that is extended to a three dimensional space, is canonically quantized. Formally one is faced with the solution of a 3-dimensional Schrödinger equation for a particle in a Coulomb potential that parametrically depends on the energy and mass integration functions of the underling cosmological model. The acceptable energy spectrum is always not empty, but in general, it is a proper subset of the mathematical energy spectrum. There is also a mass spectrum: if a mass is allowed, than also any its multiple mass is an acceptable mass. This holds also for the mass of the universe. Some results are numerically made explicit that also show the compatibility of the scheme with physical data. A universe corresponding to a proper state, is characterized by the usual Hydrogen orbitals in the physical radius. They have an internal much more complex time dependent structure when the physical radius is chosen to be solution of the underlying one dimensional Kepler equation.

**Keywords:** LTB Cosmology, Extended Quantization, Mass spectrum, Universe mass

# 1 Introduction

It is well known that the Arnold Deser Misner (ADS) decomposition of the space time metric in terms of lapse function and shift vector is the basis for an Hamiltonian formulation of General Relativity [12]. Accordingly it is possible to perform a canonical quantization procedure to obtain the Wheeler De Witt equation [3, 13, 20].
That equation is important to formulate a unified theory of General Relativity and Quantum Mechanics, that is a quantization of the gravitational field ( [19, 7, 15, 14], for a simple application [24].

The cosmological quantization problem is in particular of interest in case of the Lemaître Tolman Bondi Cosmological (LTB) model [9, 16, 1]. That model is based on the general co-moving spherically symmetric solution of the Einstein field equation for a Universe filled with freely falling dust-like matter [8]. The model has been widely discussed in terms of the mentioned Hamiltonian formalism. The Wheeler deWitt equation has been obtained and solved. The scheme has been also discussed in connection with spherically symmetric dust collapse (e. g., [18] and references therein).

The problem of the quantization of the LTB cosmology was also proposed in an elementary alternative way in [22]. In that paper one does not quantizes the Hamiltonian following the ADS procedure. Instead one consider the Kepler like one dimensional equation in the physical radius $Y(r,t)$, derived from the Einstein field equation, that depends on two arbitrary radial integration functions, say $E(r), M(r)$ [8]. Such equation is further supposed to describe the motion of a test mass $\mu$ of arbitrary, but fixed, value. Then one quantizes such equation a la Schrödinger. One is then left with a Hydrogen like atom quantum equation on the half line. By the further assumption that $E(r)$ represents the energy $W(r)$ of the Hydrogen atom, the eigenfunctions corresponding to a discrete (negative) $W(r)$ value of the energy spectrum of the H atom, represent "confined" Universes and those corresponding to positive $W(r)$ represent "not confined" Universe solutions. The image of the Universe as described by the probability distribution of the position of a test mass $\mu$ is therefore completely spherical with radial profile modulated by the behavior of Laguerre Polynomials.

There is criticism to that result. The spherical symmetry of the solution is not able to describe non spherically symmetric matter distribution of the dust matter. Also an interpretation for which the model could be suitable to represent galaxy structures, seems to work only for spherically symmetric galaxies.

In the present paper the scheme of [22] is reviewed and extended in order to possibly overcome those problems.

The procedure is the following. The mentioned Kepler like one dimensional gravitational equation in the physical radius $Y$ is extended and canonically quantized in a 3-dimensional euclidean space of polar coordinates $Y, \theta, \varphi$ with $\tau$ time. Formally one has the Schödinger equation of the $H$ atom in 3 dimensions. Again, the eigenfunctions corresponding to negative value of $E(r)$ (that is associated to the energy spectrum of the "H atom") represent "confined" Universes and those corresponding to positive $E(r)$ to "not confined" Universe solutions. There comes out an image of the Universe, that, when explored by the position probability distribution of the mass $\mu$, is given by the "classical" orbitals in the variable $Y$. The orbitals have an internal more complex structure in the variables $r, t$, if one choose $Y(r,t)$ to be solution of the LTB cosmological equation and $t = \tau$.

In case of confined universe ($E < 0$), the functions $E, M$ are no more independent so that the underline LTB cosmological model depends on one only arbitrary integration function. Moreover the "energy" spectrum is always not empty. In general it is a subset of the complete Hydrogen like mathematical spectrum. This is due to the fact that $E$ is subjected to a constraint depending on th physical role it plays in the LTB cosmological model.

The form of the universe energy spectrum has another consequence. Given a mass, then any integer multiple of that mass is a possible mass. This hold in particular for the mass of the universe, (that can be numerically evaluated), and furnishes therefore something like a mass spectrum of universe.

In spite of the present oversimplified cosmological model there is at least a compatibility of the results of the present quantization scheme with the "experimental data". This follows from the fact that the definition of mass within spherical region, induced by the present theory, gives lower values (for $E < 0$) with respect to those obtained by the conventional definition of mass.

## 2 Assumptions and preliminary considerations

The LTB cosmological model is based on the general spherically symmetric space time whose metric tensor $g_{\mu\nu}$ is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - e^{\Gamma(r,t)} dr^2 - Y^2(r,t)(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{1}$$

It is governed by the Einstein field equation [9, 16, 1, 8]:

$$R_{\mu\nu} - \frac{1}{2} R_\alpha^{\ \alpha} g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4} \tag{2}$$

$$T_{\mu\nu} = \eta(t,r) \, U_\mu U_\nu, \quad U_t = 1, \; U_k = 0, \;\; k = r, \theta, \varphi \tag{3}$$

$G$ the gravitational constant, $\eta$ the energy density of the universe, that is assumed to describe a Universe filled with dust like matter without pressure. (it is supposed the use of cgs unit system plus specified electric charge )

The $t, r$ component of the Einstein field equation can be easily integrated, with radial integration constant $E(r)$. The other component equations can be re-adjusted and further integrated, with integration constant $M(r)$, so that the scheme can be finally equivalently reduced to the set of equations [8] :

$$\frac{\dot{Y}^2}{2} - \frac{M(r)}{Y} = E(r) \tag{4}$$

$$\exp\Gamma = \frac{Y'^2(r,t)}{1+2E(r)}, \qquad M(r) = \frac{\kappa}{2}\int_0^r drY'Y^2\eta \tag{5}$$

with $\dot{Y} = \partial Y/\partial t$, $Y' = \partial Y/\partial r$

The equation (4) formally corresponds to a Kepler-like equation for the mass $M$ interacting with a test mass of value 1. We consider now the extension of the eq. (4) to the three dimensional euclidean space of coordinates $(X_1, X_2, X_3)$ and time $\tau$ for an arbitrary but fixed "test" mass $\mu$. The assumed Lagrangian and Hamiltonian are given by (now $\dot{X} = \partial_\tau$):

$$L = \frac{1}{2}\mu(\dot{X}_1^2 + \dot{X}_2^2 + \dot{X}_3^2) - U(X), \quad H = \frac{\mathbf{P}^2}{2\mu} + U(X), \quad P_k = \frac{\partial L}{\partial \dot{X}_k} \tag{6}$$

By canonical quantization one has then the Schrödinger equation:

$$\widehat{H}\psi(X,t) = i\hbar\frac{\partial\psi(X,t)}{\partial t,}, \qquad \widehat{H} = \frac{\widehat{\mathbf{P}}^2}{2\mu} + U(\widehat{X}), \tag{7}$$

where $\widehat{P}_k = -i\hbar\partial_{X_k}$, $\widehat{X} = X\cdot$. The wave function solution easily factorizes $\psi(X,t) = \phi(X)\exp(-iW\tau/\hbar)$ where $W$, the separation constant, represents the possible energy values of the model. One is then left with the eigenvalue problem

$$\Big[-\frac{\hbar^2}{2\mu}\Delta_{(X)} + U(X)\Big]\phi(X) = W\phi(X) \tag{8}$$

In order to have a quantization of the LTB cosmological model it is assumed that:

i) the coordinate $(X_1, X_2, X_3)$ are chosen so that their polar spherical representation is given by $(Y, \theta, \varphi)$;

ii) the energy potential $U(X)$ is given by $U(X) = -\frac{\mu M}{Y}$,

then $\psi(X,t) \equiv \psi(Y,\theta,\varphi,t)$ is interpreted to represent the wave function of the Universe. One is then faced with the study of the eigenvalue problem of a Hydrogen like operator. By making explicit the equation (8) in the chosen polar coordinates, [2, 11], one has

$$\Big[-\frac{\hbar^2}{2\mu}\frac{1}{Y^2}\partial_Y(Y^2\partial_Y) + \frac{\mathbf{M}^2}{2\mu Y^2} - \frac{\mu M(r)}{Y}\Big]\phi = W(r)\phi \tag{9}$$

$\mathbf{M}^2$ the square of the orbital angular momentum in the polar coordinates. By setting $\phi(Y,\theta,\varphi)=\chi(\theta,\varphi)R(Y)$ and by applying the standard quantum mechanics mathematical requirement there follows that $\chi=Y_{l,m}(\theta,\varphi)$, $Y_{l,m}$ the spherical harmonics. One is then left with the separated radial equation

$$\Big(-\frac{\hbar^2}{2\mu}\frac{d^2}{dY^2}-\frac{\mu M(r)}{Y}+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{Y^2}\Big)y(Y)=W(r)\;y(Y),\quad R(Y)=\frac{y(Y)}{Y} \tag{10}$$

that is the conventional form of the Schrödinger radial equation of the Hydrogen atom in polar coordinates whose energy spectrum is well known:

$$W_n(r)=-\frac{\mu\Big(\mu M(r)\Big)^2}{2\hbar^2}\frac{1}{n^2}=-\frac{\mu^3}{2\hbar^2}\Big(\frac{G}{c^4}\Big)^2\frac{m^2(r)}{n^2},\quad n=1,.2,3,... \tag{11}$$

From the very definition of $M$, it has been set $m(r)=\frac{c^4}{G}M=4\pi\int_o^\infty drY^2Y'\eta$. The corresponding proper solutions of (9) are:

$$\phi_{nlm}(Y,\theta,\varphi)=-\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}}\;a_n^{\frac{3}{2}}\,e^{-a_nY}\,(a_nY)^l\,L^{2l+1}_{l+m}(a_n\,Y)\,Y_{lm}(\theta,\varphi) \tag{12}$$

$$a_n=\frac{2\mu^2M(r)}{\hbar^2}\;\frac{1}{n},\quad n=1,..;\;l=0,..,n-1;\;m=-l,...,-1,..0,1,..,l \tag{13}$$

with $L^{2l+1}_{l+m}$ the Laguerre polynomials (e. g.[2, 11]).

iii) It is now further assumed the eigenvalue $W$ of eq. (8) to be related to the integration function $E$ of the LTB cosmology by $W(r)=\mu E(r),\mu>0$.

If now $E<0$, from iii) and (11), one only between $E(r)$ and $M(r)$ is an arbitrary independent function.
From (1), (5), there is the constraint $1+2E>0$ or $|W_n|<1/(2\mu)$, that is always satisfied for sufficient large $n$. Therefore the acceptable discrete spectrum is a non empty subset of the set of values in (11). The interpretation of the general solution $\psi$ of (7),

$$\psi(Y,\theta,\varphi,\tau)=\sum_{nlm}c_{nlm}e^{-\frac{i}{\hbar}W_n\tau}\,\phi_{nlm}(Y,\theta,\varphi) \tag{14}$$

is such that the wave function of the universe is superposition of "universe pure states" $\phi_{nlm}$, $|c_{nlm}|^2$ is related to the probability of the occurrence of a pure state universe $\phi_{nlm}$ with energy $W_n$. In the variables $(Y,\theta,\varphi)$, such probabilities have therefore the form of "orbitals", like those of the Hydrogen atom. In the previous calculations, $t$ and $\tau$ have been considered as disentangled variables. If one identifies $t$ and $\tau$, one can express the final results in terms of the coordinates of the LTB cosmological model by considering $Y=Y(t,r)$ to be the solution (e.g., [8] ) of the classical cosmological equation (4) with $E$ arbitrary and $M$ the corresponding expression in (11). In such case the "orbitals" have a much more complex structure in the $r$ variable and may also depend on $t$. It is worth noting that the universe state function $\psi_{nlm} \propto \exp(\frac{i}{\hbar}W_n)\phi_{nlm}$ could be suitable to describe single galaxy universe.

Analogously if $E > 0$, $W(r) = \mu E(r) > 0$ so that $1+2E > 0$. The solutions of equation (10) are then formally the scattering solutions of the $H$ atom and $E, M$ now independent functions. The universe state $\psi$ can be given as a continuous superposition in $W$ of the universe states $\phi_{Wlm}(Y, \theta, \varphi)$'s. In this case an interpretation similar to the one just mentioned for the proper state solution can be given according to the canonical interpretation of improper states in quantum mechanics applied to the present case.

# 3 Numerical considerations on the mass of the Universe

The previous results are suitable for some elementary further numerical considerations concerning the mass the universe. As it appears from the very definition, $m(r)$ is the mass contained in a sphere of radius $Y$. It is useful to possibly compare the results obtained with existing evaluations of the mass of the Universe. As it appears from the very definition, $m(r)$ is the mass contained in a sphere of radius $r$:

$$m(r) = 4\pi \int_0^r dr\, Y^2 Y' \eta(r,t) \tag{15}$$

If $m(r)$ is a non decreasing function of $r$, the $lim_{r\to\infty} m(r) = m_u$ (that is supposed to exist), can be interpreted as the total mass contained in the Universe.

Suppose now to consider the closed Universe case $E(r) < 0$ of the previous Section, assume the mass of the Universe to be given by the two possible values

$$m_u \cong 1.7 \cdot 10^{53} Kg \quad m_u \cong 6 \cdot 10^{51} Kg \tag{16}$$

where he first value in (16) of $m_u$ is taken as a mean value of some recently reported values [21, 17, 10, 5, 4] even if not all of them refer to the same definition of the mass of the Universe adopted here. If $E < 0$, the interest is here to see numerically when the entire spectrum (11) is acceptable. For this it is sufficient $|W_1| < \frac{\mu}{2}$ or

$$\frac{\mu^2}{2\hbar^2}\Big(\frac{G}{c^4}\Big)^2 m_u^2 < \frac{1}{2} \quad \Longrightarrow \quad m_u < \frac{\hbar c^4}{\mu G} \approx \frac{1}{\mu} \cdot 1.28\, 10^{22} Kg \tag{17}$$

To have compatibility of the two value in (16) with the constraint (17) it must be then respectively:

$$\mu < 7.5 \cdot 10^{-32}\, gr \cong 41\ eV, \qquad \mu < 2.13 \cdot 10^{-30} gr \cong 1.17 \cdot 10^3 eV \tag{18}$$

There follows that the neutrinos with estimated mass not greater then 0.8 eV [6], fit the mentioned constraint. Also the existence of particles of greater mass subjected to the limitations (18) is not prevented. Moreover, as already noted, the existence of higher discrete energy levels for every choice of $m_u$ with sufficiently large $n$ is always possible.

It is worth noting that the quantization of the cosmological model produces a discrete "universe mass spectrum". To see this one can use the arbitrariness of one of $M(r), E_n(r)$ in (11). For two universe masses $m_u, m'_u$, the eq. (11) can be written $E_n = C\, m_u^2/n^2,\ E'_j = Cm'^2_u/j^2,\ n, j = 1, 2, 3, ...,$ $C$ constant. Choosing $E'_1 = E_n$ then $m_u = n \cdot m'_u$. Then, by taking $r = \infty$ in (11), one has for every given $E(\infty) \equiv E_u$:

$$m_u = \frac{\sqrt{2}\hbar}{\mu}\frac{c^4}{G}\sqrt{|E_u)} \cdot n \ = 1.8\ 10^{22} \cdot \frac{\sqrt{|E_u|}}{\mu} \cdot n\ \ Kg \qquad n = 1, 2, ... \tag{19}$$

($E_u$ such that $1 + 2E_u > 0$). This could be interpreted as a universe mass quantization.

# 4 Comments

In the previous Sections the procedure of quantization of the LTB cosmological model [22], based on the quantization of the associated one dimensional Kepler like cosmological equation, has been extended to a three dimensional formulation. The formulas has been given by the correct dimensional expression in the c. g. s. unit system. This was useful for elementary numerical evaluation of the masses involved and in particular the universe mass. In this connection the mass of the universe $m_u$ here employed is that in (15), in the limit of $r \to \infty$ , but the numerical values obtained from the literature often refer to the definition of mass $m^u$ of universe so that:

$$m^u = 4\pi \int_0^\infty dr\ \eta(r,t)|g|^{\frac{1}{2}} = 4\pi \int_0^\infty dr \frac{Y^2 Y' \eta(r,t)}{\sqrt{1+2E}} \tag{20}$$

$$m^u \geq m_u \quad if \quad E(r) < 0 \tag{21}$$

$$m^u \leq m_u \quad if \quad E(r) > 0 \tag{22}$$

$g$ the determinant of the metric tensor. Therefore the present quantization scheme with $E < 0$ is compatible with "experimental data" at least for what concerns the finiteness of the universe mass.

The present method of quantization can be applied to the LTB cosmology with cosmological term. Indeed, an integration of the associated Einstein field equation gives again one dimensional Kepler equation of the form (4) with the adjoint of a harmonic potential. Such equation was quantized in [22]. It is not difficult to see that that model can be extended to a three dimensional formulation that can be quantized in a completely similar way to that of the present paper.

Finally the quantization method here proposed seems to be applicable to more complex situation. This could be the case of the LTB gravity canonically coupled to a scalar field $\phi$. Indeed, in such cases, one has again a generalized Kepler like equation (4), with however $M = M(t, r)$, plus other terms depending on $Y$ and $\phi$ [23, 25, 26].

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