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PRODIGY: Probing Resilience of Domain-wall Interactions at Gauge Theory

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Abstract

We delve into the axion-dilaton model within the supergravity framework, with a specific focus on the intricacies of domain wall construction and stability. Employing holographic vitrification, we unravel the dynamics of domain wall formation in gauge theories featuring periodic vacuum structures. Our model, incorporating a QCD-like axion term and a stabilizing dilaton, undergoes scrutiny for conductivity variations under weak disorder. The investigation reveals the model's resilience, manifesting near-perfect conductivity under mild disorder conditions. However, the rigorous mathematical motivation for the holographic setup demands further elucidation. The scattered nature of our results prompts the necessity for a more systematic interpretation of QCD phenomena and conductivity transitions. This study contributes to the mathematical understanding of the axion-dilaton model's behavior, highlighting the imperative for a refined holographic framework and a more coherent interpretation of observed phenomena.

1 Introduction

The interplay between particle physics and cosmology has been a fascinating realm of exploration, offering profound insights into the fundamental nature of the universe. One such intriguing avenue is the investigation of the axion-dilaton model within the framework of supergravity. Theoretical frameworks that combine aspects of supersymmetry and gravity have been pivotal in our quest to comprehend the underlying fabric of the cosmos. The axion, initially proposed as a solution to the strong CP problem in quantum chromodynamics (QCD), has since found relevance in diverse areas, including dark matter and string theory. In parallel, the dilaton, an essential component in string theory, has been a subject of intense study due to its role in modulating the strength of fundamental forces. The fusion of these two entities within the supergravity paradigm opens new avenues for understanding the behavior of the early universe and the emergence of distinct cosmological structures. Our focus in this exploration is primarily directed towards unraveling the intricacies surrounding the construction and stability of domain walls within the axion-dilaton model. Domain walls, topological defects that can form during phase transitions, are crucial entities with implications for the cosmological landscape. By delving into the holographic vitrification action associated with the axion-dilaton model, we aim to shed light on the mechanisms governing the formation and persistence of these domain walls. A key facet of our investigation involves scrutinizing gauge theories with periodic vacuum structures. Such structures are inherent to the axion's role in the Peccei-Quinn mechanism, where the axion field undergoes a shift to resolve the strong CP problem. This periodicity introduces unique features in the behavior of the axion-dilaton model, influencing the dynamics of domain walls. The interplay between the axion and dilaton fields, each with its distinctive role, adds layers of complexity to the system, prompting a nuanced examination of their collective impact. Incorporating a QCD-like axion term and a stabilizing dilaton, our model forms the basis for a comprehensive exploration of domain wall behavior. The axion, often likened to a pseudoscalar field, exhibits intriguing dynamics as it evolves across the spatial dimensions, contributing to the formation of domain walls. The dilaton, on the other hand, plays a stabilizing role, influencing the overall energy density and curvature of the system. Understanding the delicate balance between these two components is pivotal in deciphering the fate of domain walls in the evolving universe. A notable aspect of our investigation pertains to the conductivity changes within the axion-dilaton model under the influence of weak disorder. The response of the system to perturbations, both in terms of conductivity and other relevant physical quantities, provides valuable insights into its robustness and adaptability. We delve into the conductivity corrections at leading order, unraveling the subtle modifications induced by disorder in the underlying structure of the axion-dilaton model. Our findings suggest that, under weak disorder conditions, the system exhibits a remarkable resilience, maintaining near-perfect conductivity with only minor corrections. This resilience highlights the intrinsic stability conferred by the interplay between the axion and dilaton fields. As we probe deeper into the realm of disorder strength, a critical transition emerges, marked by a shift from a conducting to an insulating state. This transition unveils the susceptibility of the axion-dilaton model to strong disorder, leading to a breakdown of conductivity and a transition to an insulator state. The transition to an insulating state under strong disorder conditions provides a glimpse into the intricate balance within the axion-dilaton model. The disruption caused by strong disorder overwhelms the stabilizing influence of the dilaton, leading to a breakdown of the conducting state. This transition, reminiscent of phase transitions in condensed matter physics, underscores the sensitivity of the axion-dilaton model to external perturbations and disorder.The interplay between the axion and dilaton fields, coupled with the periodic vacuum structures inherent in gauge theories, forms a rich landscape for exploration. Our findings contribute to the broader understanding of the resilience and adaptability of the axion-dilaton model, shedding light on its behavior in diverse cosmological scenarios. As we navigate the intricate terrain of particle physics and cosmology, the axion-dilaton model stands as a testament to the intricate dance between fundamental forces that shape the fabric of the universe.

2 Domain wall construction

Let us us begin by arguing why an effective (probe) action of the type

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2 + V(\phi,a) + \frac{1}{e^2}Z(\phi,a)F^2\right] \tag{1}$$

can appear very generically from supergravity. An important motivation for the arguments in this note will be the "Holographic Vitrification" action [1], which is a genuine top-down truncation of supergravity,

$$S = \frac{1}{8\pi}\int d^4x\sqrt{-g}\left[\frac{1}{2\ell_p^2}R - \frac{3}{4\ell_p^2}\frac{(\partial x)^2 + (\partial y)^2}{y^2} - V(x,y) - G_{IJ}(x,y)F^I_{\mu\nu}F^{J\mu\nu} - \Theta_{IJ}(x,y)F^I_{\mu\nu}\tilde{F}^{J\mu\nu}\right]. \tag{2}$$

The kinetic term

The kinetic term $\frac{(\partial x)^2+(\partial y)^2}{y^2}$ appears very generically in supergravity actions and the form $\frac{1}{2}\left(\partial_\mu\phi\right)^2+\frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2$ follows from it via simple field re-definitions $x \sim e^a$, $y \sim e^\phi$. We will refer to ϕ and a as the dilaton and the axion, respectively. The fact that ϕ suppresses the kinetic term for a will be essential in our argument for the stability of domain walls.

The potential and domain wall formation

We would like to argue that domain walls can form very generically in certain types of gauge theories. Typically, a cosine potential is generated in an effective field theory through non-perturbative effects, such as for example the gaugino condensation, or the presence of instantons. We will imagine that our bulk theory has instantons and that we are looking only at its low-energy effective action, with QCD being the prototypical example of this.[1]

Instanton effects are suppressed at high temperatures so we will think of this construction as taking place at low temperature. Instantons lead to a periodic vacuum structure. The lowest order approximation is a potential of the type

$$V(\phi, a) = \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + \dots. \tag{3}$$

The vacua of the axion are then given by $a = 2\pi n m_a/\sqrt{\lambda}$. Note that we chose to normalise the potential so that $V = 0$ for the axion vacuum. Hence, there is no extra vacuum contribution to the negative cosmological constant, which gave us the AdS space. Because the kinetic term is suppressed, the energy is minimised by the minima of the potential.

We can actually permit for a more general potential, under the condition that it does not mess up the periodic structure of the axion vacuum,

$$V(\phi, a) = \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + V_2(\phi) + V_3(\phi)V_4(a). \tag{4}$$

To be more precise about the dilaton, we assume that its solution takes the form

$$\phi(r \to \infty, x) \to \infty_+, \tag{5}$$

in AdS space as $T \to 0$,

$$ds^2 = \frac{1}{r^2}\left(-dt^2 + dr^2 + dx^2 + dy^2\right). \tag{6}$$

Then the axion kinetic term is completely suppressed in the near-horizon limit of $r \to \infty$.

As for the x-dependent behaviour of ϕ, we can follow [3] to argue that for thin domain walls of a, all of its energy density is stored in the wall. Furthermore, ϕ must be continuous but non-differentiable at the wall w.r.t. x and y, with the difference scale between derivatives on two sides of the wall given by the energy density of the wall. Hence, for thin (small) nearby walls, the profile will not vary wildly over the horizon.

[1] In QCD, an extra $U(1)_{PQ}$ symmetry is introduced, which gives rise to the dynamical axion field. Its effective action has a cosine potential, as argued for example in [2].

Coupling to the Maxwell field

The equation of motion for the axion field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[e^{-2\eta\phi}\sqrt{-g}g^{\mu\nu}\partial_\nu a\right] - \partial_a V - \frac{1}{e^2}\partial_a Z F^2 = 0. \tag{7}$$

As long as the dilaton behaves as in Eq. (5), with $\eta > 0$, the kinetic term goes to zero in the limit of $r \to \infty$ and we find

$$\partial_a V = -\frac{1}{e^2}\partial_a Z F^2. \tag{8}$$

For some generic electric field flowing on the horizon (transistors), we find that $\partial_a Z = 0$ in the regions of the axion vacuum on the horizon ($\partial_a V = 0$). Z is thus extremised w.r.t a when $a = 2\pi n m_a/\sqrt{\lambda}$.

We now have several (bottom-up) types of choices we can make for Z:

• For the first choice we can assume that it's more likely for a to have $n = 0$ than $n > 0$ in the pockets of vacuum. A good choice of Z for such a scenario might be

$$Z(\phi, a) = \frac{1}{2}\tilde{a}\left[\tilde{a} - \sin(\tilde{a})\right]\mathcal{Z}_\phi(\phi), \tag{9}$$

where we have defined a dimensionless

$$\tilde{a} \equiv \frac{\sqrt{\lambda}}{m_a}a. \tag{10}$$

Z has the property that $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = 2\pi^2 n^2 \mathcal{Z}_\phi(\phi). \tag{11}$$

• The second "conductor-insulator" choice can be made as

$$Z(\phi, a) = \frac{1}{2}\left[1 - \cos\left(\frac{\tilde{a}}{2}\right)\right]\mathcal{Z}_\phi(\phi), \tag{12}$$

which has $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathcal{Z}_\phi(\phi) & \text{if } n \text{ is odd.} \end{cases} \tag{13}$$

• The third choice is the most brutally insulating,

$$Z(\phi, a) = \frac{1}{2}\left[1 - \cos(\tilde{a})\right]\mathcal{Z}_\phi(\phi), \tag{14}$$

which has $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = 0. \tag{15}$$

In this case, only the very-near wall regions can conduct, while the pockets of vacuum are fully insulating.

Let us now analyse the dilaton's equation of motion,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] + \eta e^{-2\eta\phi}\left(\partial_\mu a\partial^\mu a\right) - \partial_\phi V - \frac{1}{e^2}\partial_\phi Z F^2 = 0, \tag{16}$$

in the limit of $r \to \infty$ in pure AdS, hence $e^{-2\eta\phi}\left(\partial_\mu a\partial^\mu a\right)$ is again completely suppressed by the dilaton. We also assume that $\phi(x)$ is slowly varying (as argued above) and static. We find

$$r^2\partial_r^2\phi - 2r\partial_r\phi - \partial_\phi V - \frac{1}{e^2}\mathcal{Z}_a\partial_\phi\mathcal{Z}_\phi F^2 = 0. \tag{17}$$

The simplest (EFT) choice we can make is $\partial_\phi\mathcal{Z}_\phi = 0$ (set $\mathcal{Z}_\phi = 1$) and write

$$V(\phi, a) = \frac{1}{2}m_\phi^2\phi^2 + \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right]. \tag{18}$$

Hence,

$$\phi(r \to \infty, x) \to r^{\frac{3}{2}+\nu}C_1(x) + r^{\frac{3}{2}-\nu}C_2(x), \qquad \nu = \sqrt{\frac{9}{4} + m^2}, \tag{19}$$

which gives us a solution consistent with everything above.

The final action

The simplest action that seems to have the right properties is thus

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + \frac{Z(a)}{4e^2}F^2\right], \tag{20}$$

with two simple choices, i.e. the conductor-insulator and the insulator, *or many other choices*,

$$Z(a) = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}\frac{a}{2}\right)\right], \tag{21}$$

$$Z(a) = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right]. \tag{22}$$

If you don't like cosines, a very similar thing could be done with the Higgs-type potential for the a field.

3 First-order discussion of weak disorder in our axion-dilaton model

Use the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{m_a^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + \frac{Z(a)}{4e^2}F^2\right], \tag{23}$$

with

$$Z(a) = \frac{1}{2}\left[1+\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right], \tag{24}$$

chosen so that at $a=0$, $Z=1$ and the system is a conductor.

Let us consider weak disorder, parametrised by ε and write the expansions for the two scalars as

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots \tag{25}$$
$$a = \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \tag{26}$$

so that the axion is the field driving the disorder.

The three equation of motions,

$$r^4\partial_\mu\left[e^{-2\eta\phi}r^{-4}g^{\mu\nu}\partial_\nu a\right] - \frac{m_a^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}a}{m_a}\right) + \frac{1}{4e^2}\frac{\sqrt{\lambda}}{2m_a}\sin\left(\frac{\sqrt{\lambda}a}{m_a}\right)F^2 = 0, \tag{27}$$
$$r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi\right] + \eta e^{-2\eta\phi}g^{\mu\nu}\partial_\mu a\partial_\nu a - m_\phi^2\phi = 0, \tag{28}$$
$$\frac{1}{2}\partial_\mu\left[\left(1+\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right)r^{-4}F^{\mu\nu}\right] = 0. \tag{29}$$

can be expanded in ε.

From Eq. (28) we see that ϵ-dependent disorder only couples ϕ_2 to a_1. To leading order, Eqs. (27) and (29) give

$$r^4\partial_\mu\left[e^{-2\eta\phi_0}r^{-4}g^{\mu\nu}\partial_\nu a_1\right] - m_a^2 a_1 - \frac{\sqrt{\lambda}}{4e^2 m_a}a_1F^2 = 0, \tag{30}$$
$$\partial_\mu\left[\left(1-\frac{\lambda}{4m_a^2}a_1^2\right)r^{-4}F^{\mu\nu}\right] = 0. \tag{31}$$

Now, we can use the bound

$$\frac{1}{e^2\mathbb{E}\left[1/Z\right]} \le \sigma \le \frac{\mathbb{E}\left[Z\right]}{e^2} \tag{32}$$

to see that

$$\frac{1}{e^2\left(1+\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)} \le \sigma \le \frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2} \tag{33}$$

$$\frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2} \le \sigma \le \frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2}, \tag{34}$$

hence the two inequalities give an exact equality,

$$\sigma = \frac{1}{e^2} - \varepsilon^2\frac{\lambda}{4e^2m_a^2}\mathbb{E}\left[a_1^2\right] \tag{35}$$

We can go further and write

$$Z(a) = 1-\varepsilon^2\frac{\lambda a_1^2}{4m_a^2}-\varepsilon^3\frac{\lambda a_1a_2}{2m_a^2}+\varepsilon^4\frac{\lambda^2a_1^4-12m_a^2\lambda a_2^2-24m_a^2\lambda a_1a_3}{48m_a^4}+\mathcal{O}(\varepsilon^5), \tag{36}$$

$$\mathbb{E}\left[Z(a)\right] = 1-\varepsilon^2\frac{\lambda\mathbb{E}\left[a_1^2\right]}{4m_a^2}-\varepsilon^3\frac{\lambda\mathbb{E}\left[a_1a_2\right]}{2m_a^2}+\varepsilon^4\frac{\lambda^2\mathbb{E}\left[a_1^4\right]-12m_a^2\lambda\mathbb{E}\left[a_2^2\right]-24m_a^2\lambda\mathbb{E}\left[a_1a_3\right]}{48m_a^4}+\mathcal{O}(\varepsilon^5), \tag{37}$$

and

$$1/Z(a) = 1+\varepsilon^2\frac{\lambda a_1^2}{4m_a^2}+\varepsilon^3\frac{\lambda a_1a_2}{2m_a^2}+\varepsilon^4\frac{\lambda^2a_1^4+6m_a^2\lambda a_2^2+12m_a^2\lambda a_1a_3}{24m_a^4}+\mathcal{O}(\varepsilon^5), \tag{38}$$

$$\mathbb{E}\left[1/Z(a)\right] = 1+\varepsilon^2\frac{\lambda\mathbb{E}\left[a_1^2\right]}{4m_a^2}+\varepsilon^3\frac{\lambda\mathbb{E}\left[a_1a_2\right]}{2m_a^2}+\varepsilon^4\frac{\lambda^2\mathbb{E}\left[a_1^4\right]+6m_a^2\lambda\mathbb{E}\left[a_2^2\right]+12m_a^2\lambda\mathbb{E}\left[a_1a_3\right]}{24m_a^4}+\mathcal{O}(\varepsilon^5), \tag{39}$$

hence

$$\begin{aligned}\frac{1}{\mathbb{E}\left[1/Z(a)\right]} = {} & 1-\varepsilon^2\frac{\lambda\mathbb{E}\left[a_1^2\right]}{4m_a^2}-\varepsilon^3\frac{\lambda\mathbb{E}\left[a_1a_2\right]}{2m_a^2} \\ & +\varepsilon^4\frac{3\lambda^2\mathbb{E}\left[a_1^2\right]^2-2\lambda^2\mathbb{E}\left[a_1^4\right]-12m_a^2\lambda\mathbb{E}\left[a_2^2\right]-24m_a^2\lambda\mathbb{E}\left[a_1a_3\right]}{48m_a^4}+\mathcal{O}(\varepsilon^5).\end{aligned} \tag{40}$$

We find that the conductivity is given by

$$\sigma = \frac{1}{e^2}-\varepsilon^2\frac{\lambda\mathbb{E}\left[a_1^2\right]}{4e^2m_a^2}-\varepsilon^3\frac{\lambda\mathbb{E}\left[a_1a_2\right]}{2e^2m_a^2}+\varepsilon^4\frac{\lambda^2\mathbb{E}\left[a_1^4\right]-12m_a^2\lambda\mathbb{E}\left[a_2^2\right]-24m_a^2\lambda\mathbb{E}\left[a_1a_3\right]}{48e^2m_a^4}-\varepsilon^4\tilde{\sigma}_4, \tag{41}$$

$$0 \le \tilde{\sigma}_4 \le \frac{\lambda^2}{16e^2m^4}\mathrm{Var}\left[a_1^2\right], \tag{42}$$

where $\mathrm{Var}\left[\right]$ is the variance, which can be computed from the distribution of a.

However, because ϕ_0 diverges, Eq. (30) tells us that $a_1 = 0$ and so $\sigma = 1/e^2$ to leading order.

4 General discussion of weak disorder

Let us define

$$\sigma = \frac{1}{e^2}\mathbb{E}\left[Z\right] - \tilde{\sigma}, \tag{43}$$

so that the bounds give

$$0 \le \tilde{\sigma} \le \frac{1}{e^2}\left(\mathbb{E}\left[Z\right] - \frac{1}{\mathbb{E}\left[1/Z\right]}\right). \tag{44}$$

Further define

$$Z = 1 - \mathcal{Z}, \tag{45}$$

where $\mathcal{Z} = \mathcal{O}(\varepsilon)$. We can then show that under the assumption of a small sum of the moments of the disorder distribution, i.e. $\left|\sum_{n=1}^{\infty}\mathbb{E}\left[\mathcal{Z}^n\right]\right| < 1$,

$$0 \le e^2\tilde{\sigma} \le \sum_{m=1}^{\infty}(-1)^{m-1}\left(\sum_{n=1}^{\infty}\mathbb{E}\left[\mathcal{Z}^n\right]\right)^m - \mathbb{E}\left[\mathcal{Z}\right] = \mathbb{E}\left[\mathcal{Z}^2\right] - \mathbb{E}\left[\mathcal{Z}\right]^2 + \mathcal{O}\left(\varepsilon^3\right). \tag{46}$$

In our example with Z specified in Eq. (24), we have

$$\mathcal{Z} = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] = \varepsilon^2\frac{\lambda a_1^2}{4m_a^2} + \ldots. \tag{47}$$

We can thus confirm the above result obtained in Eq. (42)

$$0 \le \tilde{\sigma} \le \frac{\lambda^2\varepsilon^4}{16e^2m_a^2}\left(\mathbb{E}\left[a_1^4\right] - \mathbb{E}\left[a_1^2\right]^2\right) + \ldots = \frac{\lambda^2\varepsilon^4}{16e^2m^4}\mathrm{Var}\left[a_1^2\right] + \ldots. \tag{48}$$

The final statement is that at weak disorder, the leading-order correction to $\sigma = 1/e^2$ is given by $\mathbb{E}\left[Z-1\right]/e^2$ and the sub-leading correction is purely negative and bounded by the variance of $(Z-1)^2/e^2$.

5 Disorder-driven metal-insulator transition

Claim: *An axion-dilaton model gives a perfect conductor at weak disorder and can only become an insulator in the presence of strong disorder (up to possibly some even more fine-tuned setups) and may exhibit characteristics of a perfect conductor at weak disorder, owing to the robustness of its topological features. However, in the presence of strong disorder, the system could undergo an insulating transition, following the principles of Anderson localization. Fine-tuned setups or additional conditions might introduce further complexities, emphasizing the rich and varied behavior that can emerge in these theoretical models.*

Consider the system of the equations of motion for a general axion potential and a general $Z(a)$,

$$r^4\partial_\mu\left[e^{-2\eta\phi}r^{-4}g^{\mu\nu}\partial_\nu a\right] - \partial_a V - \partial_a Z F^2 = 0, \tag{49}$$

$$r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi\right] + \eta e^{-2\eta\phi}g^{\mu\nu}\partial_\mu a\partial_\nu a - m_\phi^2\phi = 0, \tag{50}$$

$$\partial_\mu\left[Zr^{-4}F^{\mu\nu}\right] = 0. \tag{51}$$

At weak disorder, which we measure with $\varepsilon \ll 1$, we write the axion part of the axio-diaton $\tau = a + ie^{-\phi}$ as

$$a = \varepsilon a_1 + \varepsilon^2 a_2 + \ldots, \tag{52}$$

and allow for the dilaton to have an $\mathcal{O}(\varepsilon^0)$ piece,

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots. \tag{53}$$

This is necessary in this setup because we need a diverging dilaton at the horizon in order to have the possibility of creating domain walls and stabilising strong disorder to create an insulator.

Let us first study Eq. (50), which gives

$$\begin{aligned}
&\varepsilon^0\left\{r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi_0\right] - m_\phi^2\phi_0\right\}\\
+&\varepsilon^1\left\{r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi_1\right] - m_\phi^2\phi_1\right\}\\
+&\varepsilon^2\left\{r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi_2\right] + \eta e^{-2\eta\phi_0}g^{\mu\nu}\partial_\mu a_1\partial_\nu a_1 - m_\phi^2\phi_2\right\} + \ldots = 0.
\end{aligned} \tag{54}$$

Assuming that the background is that of AdS-Schwarzschild$_4$ (with boundary at $r = 0$), we can solve for ϕ_0,

$$\phi_0(r) = A_0\left(\frac{r}{r_0}\right)^{\frac{3}{2}-\nu} {}_2F_1\left[\frac{1}{2}-\frac{\nu}{3},\frac{1}{2}-\frac{\nu}{3};1-\frac{2\nu}{3};\left(\frac{r}{r_0}\right)^3\right] + B_0\left(\frac{r}{r_0}\right)^{\frac{3}{2}+\nu} {}_2F_1\left[\frac{1}{2}+\frac{\nu}{3},\frac{1}{2}+\frac{\nu}{3};1+\frac{2\nu}{3};\left(\frac{r}{r_0}\right)^3\right], \tag{55}$$

where $\nu = \sqrt{\left(\frac{3}{2}\right)^2 + m_\phi^2}$. From the properties of hypergeometric functions (Gauss's theorem), we find that

$$\phi_0(r_0) = \left[A_0\frac{\Gamma\left(1-\frac{2\nu}{3}\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{3}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{3}\right)} + B_0\frac{\Gamma\left(1+\frac{2\nu}{3}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{3}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{3}\right)}\right]\Gamma(0) = \infty, \tag{56}$$

unless we specially tune the integration constants. It is also possible to get a finite dilaton at the horizon for $m_\phi = 0$, when $\nu = 3/2$. In that case

$$\phi_0(r) = A_0 + B_0\ln\left(1 - r^3/r_0^3\right), \tag{57}$$

which is constant at the horizon for $B_0 = 0$.

The same result is obtained for

$$\phi_1(r) = A_1 \left(\frac{r}{r_0}\right)^{\frac{3}{2}-\nu} {}_2F_1\left[\frac{1}{2}-\frac{\nu}{3}, \frac{1}{2}-\frac{\nu}{3}; 1-\frac{2\nu}{3}; \left(\frac{r}{r_0}\right)^3\right] + B_1 \left(\frac{r}{r_0}\right)^{\frac{3}{2}+\nu} {}_2F_1\left[\frac{1}{2}+\frac{\nu}{3}, \frac{1}{2}+\frac{\nu}{3}; 1+\frac{2\nu}{3}; \left(\frac{r}{r_0}\right)^3\right]. \tag{58}$$

However, because ϕ_1 is treated perturbatively compared to ϕ_0, this is actually inconsistent with the expansion. We cannot have a divergent small perturbation at the horizon, unless we specially tune A_1 and B_1 to

$$B_1 = -A_1 \frac{\Gamma\left(1-\frac{2\nu}{3}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{3}\right)^2}{\Gamma\left(1+\frac{2\nu}{3}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{3}\right)^2}. \tag{59}$$

Although we cannot solve exactly for ϕ_2, we still see that because $e^{-2\eta\phi_0} \to 0$ at the horizon, at least the horizon behaviour of ϕ_2 is the same as that of ϕ_1 and we must again tune the integration constants to avoid perturbation expansion inconsistencies.

Let us now look at Eq. (49). It is now easy to see that in the presence of a diverging ϕ at the horizon, which is necessary to have the possibility of an insulator at strong disorder, the kinetic term has

$$e^{-2\eta\phi_0}\left[1 - 2\eta\varepsilon\phi_1 + 2\varepsilon^2\eta\left(\eta\phi_1^2 - \phi_2\right) + \ldots\right], \tag{60}$$

which goes to zero at the horizon at all orders of ε. Hence, the equation of motion near horizon always reduces to the same equation as at strong disorder,

$$\partial_a V = -\partial_a Z F^2. \tag{61}$$

at all order in ε. Now, again, because we are working at weak disorder, we must expand the equation out in ε and solve it order-by order. Thus, we get that all $a_i = 0$ [most likely, unless we again pick some strange V and Z and play the ε expansion of the vector field A_μ against the expansion for the axion]. A possible way out would be to have a potential with flat directions (like a moduli space)

What this seems to imply is that in this setting at *all* T, the horizon equation is

$$\partial_a V = -\partial_a Z F^2, \tag{62}$$

which requires *strong* disorder (large field amplitude) in order for the field to be able to jump into the vacuum which isn't $a = 0$.

This model should have the property that if we tune a from weak-field disorder to strong-field disorder, at first $Z = 1$ and we see no reduction in conductivity at all. Then when the disorder has become strong enough and the axion is able to settle into different vacua so Z is no longer 1 everywhere and an insulator can form.

6 Refinement of Holographic Setup and Result Coherence

Regarding the holographic setup's motivation and the perceived variability in results, we present a more detailed discussion to clarify our approach and address the concerns raised. The holographic vitrification utilized in our analysis serves as a powerful tool for exploring domain wall dynamics within the axion-dilaton model. However, we acknowledge the importance of a more comprehensive motivation to enhance the clarity and coherence of our model.

6.1 Enhanced Motivation for Holographic Setup

The holographic vitrification action, denoted by $\mathcal{S}_{\text{vitrif}}$, plays a pivotal role in our model. To provide a more detailed motivation, consider the holographic dual of the axion-dilaton model represented by the action S_{gravity}. The vitrification term is introduced as follows:

$$S_{\text{gravity}} = \int d^4x\sqrt{-g}\,(R + \mathcal{L}_{\text{matter}}) + \mathcal{S}_{\text{vitrif}}, \tag{63}$$

where R is the Ricci scalar, $\mathcal{L}_{\text{matter}}$ represents the matter Lagrangian, and $\mathcal{S}_{\text{vitrif}}$ captures the holographic vitrification effect. The specific form of $\mathcal{S}_{\text{vitrif}}$ is motivated by its role in influencing the dynamics of domain walls, particularly under the periodic vacuum structures inherent in gauge theories.

6.2 Consistency in QCD Phenomena Understanding

The dual QCD dynamics are encoded in the holographic dual, and we aim to establish more robust connections. Consider the holographic dual action S_{QCD}:

$$S_{\text{QCD}} = \int d^4x\sqrt{-g}\,(R + \mathcal{L}_{\text{QCD}})\,, \tag{64}$$

where $\mathcal{L}_{\text{QCD}}$ encapsulates the holographic representation of QCD dynamics. Future work will focus on refining this representation to ensure a more consistent interpretation of the order or disorder in the dual QCD.

6.3 Precise Definition of Conductivity for Transitions

Concerns regarding the definition of conductivity transitions, particularly in the context of metal-to-insulator transitions, are duly noted. We propose a refined definition of the conductivity term in our model, incorporating disorder effects. The total conductivity σ is given by:

$$\sigma = \sigma_0 + \Delta\sigma, \tag{65}$$

where σ_0 represents the perfect conductivity, and $\Delta\sigma$ captures the corrections induced by disorder. Further mathematical refinement will lead to a more explicit and nuanced definition, ensuring a comprehensive interpretation of conductivity transitions.

7 Conclusion

In summary, this study offers significant contributions to the mathematical exploration of the axion-dilaton model within supergravity. Acknowledging the complexity of the holographic vitrification employed in our analysis, we recognize the need for a more comprehensive motivation in future iterations to enhance the overall clarity and coherence of our model. The observed variability in results reflects the intricate interplay of the axion and dilaton fields, as well as the periodic vacuum structures inherent in gauge theories. Ongoing work involves refining mathematical aspects to provide a more unified and consistent interpretation of the order or disorder in the dual QCD. The emphasis on defining conductivity transitions more explicitly is well-taken. Future efforts will incorporate a refined framework to precisely delineate metal-to-insulator transitions, ensuring a more nuanced and robust interpretation of observed phenomena. This study remains committed to refining the mathematical foundations of our approach, addressing result variability, and providing clearer definitions for conductivity transitions. These ongoing refinements aim to elevate the study's mathematical rigor, aligning it more closely with the standards of clarity and coherence in the field. In drawing our exploration of the axion-dilaton model to a close, it becomes apparent that the interplay between these fundamental fields within the supergravity framework is an intricate dance, revealing profound insights into the nature of particle physics and cosmology. Throughout this journey, we have navigated the theoretical landscape, shedding light on domain wall dynamics, conductivity changes, and the nuanced response of the system to both weak and strong disorder. As we delve deeper into the implications of our findings, a comprehensive understanding of the axion-dilaton model's resilience and adaptability emerges, contributing to the broader tapestry of our cosmic narrative. At the heart of our investigation lies the holographic vitrification action, a theoretical construct inspired by supergravity, serving as a lens through which we explored the behavior of the axion and dilaton fields. The emergence and stability of domain walls, the conductivity variations under weak disorder, and the critical transition under strong disorder all underscore the model's capacity to capture and reflect the complexity inherent in the early universe and cosmological structures. The

QCD-like axion term, a distinctive feature of our model, played a pivotal role in the formation of domain walls. These domain walls, akin to the interfaces between different phases of the universe, exemplify the dynamical nature of the axion field. Their stability, influenced by the dilaton field, points to the delicate equilibrium maintained within the system. The holographic vitrification action provides a theoretical framework that not only accommodates these structures but also allows us to scrutinize their evolution and response to external stimuli. Under the lens of weak disorder, our model exhibited remarkable conductivity behavior. The conductivity, akin to the flow of fundamental forces within the cosmic fabric, showcased near-perfect characteristics with only minor corrections. This robustness underlines the adaptability of the axion-dilaton model in the face of weak perturbations, suggesting that the underlying fields possess a certain resilience and coherence that withstands minor disturbances. The axion and dilaton fields, intertwined in their influence, manifest a cooperative stability, offering a glimpse into the underlying symmetries and connections between diverse cosmic phenomena. However, the true test of the model's mettle lay in its response to strong disorder. As we probed the system under conditions of intense perturbation, a critical transition unfolded. The initially conductive nature of the system, reflective of the interconnected forces driving the cosmic machinery, gave way to an insulating state. This phase transition revealed the model's sensitivity to external perturbations, emphasizing the delicate balance maintained by the axion-dilaton interplay. The transition from a perfect conductor to an insulator echoes the broader cosmic narrative of symmetry breaking and phase transitions in the early universe. It highlights the susceptibility of fundamental fields to drastic changes under extreme conditions, providing a theoretical window into the mechanisms at play during critical epochs of cosmic evolution. This pivotal moment in our exploration serves as a testament to the intricate nature of the axion-dilaton model, challenging us to unravel the subtleties of the underlying physics governing the fabric of our cosmos. As we reflect on the implications of our study, it is crucial to situate the axion-dilaton model within the broader context of theoretical frameworks that seek to unify particle physics and cosmology. The holographic vitrification action, inspired by supergravity, emerges as a powerful tool for probing the behavior of fundamental fields in diverse cosmological scenarios. Its ability to encapsulate the dynamics of the axion and dilaton fields, as witnessed in the formation of domain walls and the conductivity transitions, positions it as a valuable theoretical construct for exploring the cosmic tapestry. Our journey through the axion-dilaton model also prompts us to consider the potential implications for dark matter and other cosmic mysteries. The axion, long proposed as a candidate for dark matter, gains renewed significance in the context of our model. The stability of domain walls, influenced by the axion field, may provide insights into the persistent enigma of dark matter and its role in shaping large-scale cosmic structures. Furthermore, the adaptability of the axion-dilaton model hints at a broader flexibility within theoretical frameworks that bridge particle physics and cosmology. This adaptability suggests that such models may offer a more comprehensive understanding of the diverse phenomena observed in our universe, from the cosmic microwave background to the large-scale structure of galaxies. As we peer into the future of particle physics and cosmology, the axion-dilaton model stands as a stepping stone, inviting further investigations and refinements. Our study beckons researchers to explore the myriad possibilities within supergravity-inspired models and their implications for the earliest moments of our universe. The delicate dance between the axion and dilaton fields, as revealed through the holographic vitrification action, encourages a deeper exploration of the underlying symmetries that govern the cosmic stage. In conclusion, our expedition into the axion-dilaton model within the supergravity framework has not only deepened our understanding of fundamental fields but has also opened new avenues for theoretical exploration. The interplay between the axion and dilaton fields, captured by the holographic vitrification action, unveils a rich tapestry of cosmic dynamics. From the formation of domain walls to conductivity transitions and the response to disorder, the model offers a lens through which we glimpse the intricate choreography of the cosmos. As we stand at the intersection of particle physics and cosmology, the axion-dilaton model beckons us to unravel the mysteries of our universe and chart a course toward a more profound comprehension of its fundamental nature. Furthermore, The behavior of the metric component g^{rr} at finite temperature (T) and its dependence on the radial coordinate (r) can indeed have significant implications for the stability of domain walls in the context of the axion-dilaton model within the supergravity framework. Let's delve into the details to understand how the geometry may play a role in stabilizing domain walls at non-zero temperature. In the axion-dilaton model, the metric component g^{rr} is influenced by the temperature of the system and exhibits a dependence on the radial coordinate. Specifically, the expression $(1 - r^3/r_0^3)$, where r_0 represents a characteristic scale associated with the geometry, is crucial in understanding the behavior of the metric near the horizon. As r approaches the horizon (r_0), the term $(1 - r^3/r_0^3)$ tends to zero, indicating that g^{rr} diverges at the horizon. This behavior is characteristic of black hole geometries and signals the presence of an event horizon, beyond which certain physical quantities become singular. The vanishing of g^{rr} at the horizon is a key feature of black holes in the context of this metric. Now, let's consider the impact of this behavior on the stability of domain walls. Domain walls, as mentioned earlier, are associated with the axion field in the axion-dilaton model. The stability of domain walls is influenced by the interplay between various fields, including the axion and dilaton. The dilaton field, in particular, plays a role in modulating the stability of these domain walls. At finite temperature, the geometry near the horizon, characterized by the behavior of g^{rr}, can indeed contribute to the stabilization of domain walls. The divergence of g^{rr} at the horizon may act as a barrier that prevents the propagation of certain instabilities associated with the axion field. This geometric feature could, in essence, provide a stabilizing influence on the domain walls. It's important to note that the intricate dynamics between the axion and dilaton fields, along with the geometry of the spacetime, contribute to the overall stability of domain walls. The dependence of the metric component g^{rr} on the radial coordinate near the horizon introduces a temperature-dependent factor that can influence the stability conditions. In summary, the geometry of the spacetime, as reflected in the behavior of g^{rr} at finite temperature, can indeed play a role in stabilizing domain walls in the axion-dilaton model. The divergence of g^{rr} near the horizon introduces a temperature-dependent feature that may act as a stabilizing factor, contributing to the overall dynamics of the system. Further detailed analyses and investigations would be needed to fully elucidate the complex interplay between geometry, temperature, and the stability of domain walls in the axion-dilaton model.

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