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On the Method of Stationary Phase in Calculating the Propagator of the Coulomb-Kepler Problem (II.)

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Abstract

The magnitudes are scaled with the aid of mean values by the initial wave function. Main scaling parameter is λ, which denotes the angular momentum in units of $\hbar$. In order to achieve compact analytic results, the dimensionless time τ is limited by the condition $0 \leq \tau \ll \lambda$. Then, as it is shown by means of a theorem on the method of stationary phase, the propagator has the structure of Hostler's Green's function [6], but with dressed magnitudes. For instance, the potential strength parameter Q_0 is replaced by Q_0^{eff} which now depends both on the Lambert variables and on time. Basically, Hostler's Green's function is the product of the two Whittaker functions. By applying the saddle-point method, the product, which now depends on dressed variables, is asymptotically approximated for large λ in terms of elementary functions. A numerical example gives agreement between the asymptotic expression and the original product in the interval $15 \leq \lambda \leq 36$, without visible deviations in the graphical plot. The approximated propagator, defined in the restricted time interval, now is in a feasible form for docking to the initial state of a rectilinear orbit, for instance.

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1 Introduction

The present paper deals with the quantum mechanics of the Coulomb-Kepler problem, and tries to go beyond successful applications of the Kustaanheimo-Stiefel transformation (KST) [7], e.g. by Gerry [4] and by [8], [9]. The KST transforms the non-relativistic Hamiltonian into a Hamiltonian of four harmonic oscillators, and offers "time" dependent solutions in terms of coherent states. Time is set in quotation marks since, by the transformed Hamiltonian, the wave function actually evolves by the eccentric anomaly w as a curve parameter rather than by the Schrödinger time t. The requirement that w and t have to be in 1-1 correspondence is quite restrictive. As is demonstrated in [10], the condition $\mathrm{d}w/\mathrm{d}t > 0$ can break down in the quantum regime at a finite time $t_c > 0$, which, in particular, is the case for initially rectilinear orbits [11].

The propagator is derived by means of Fourier transformation in complex energy space E from the Green's function G of the Coulomb-Kepler problem. G is available in a compact analytic form as derived by Hostler [6]. After a transformation of the integration path, the task is to calculate the following integral, which was shown in [12]:

$$\begin{aligned} K(\mathbf{r_1},\mathbf{r_2},t) \;&=\; \frac{\mathbf{i}}{\pi}\lim_{\eta\to 0}\left\{\int_{A_1}^{A_2} \mathrm{d}A\,A\exp[\mathbf{i}\,A^2\,t/\hbar]G_E(\mathbf{r},\mathbf{r}')\right\}, \qquad (1)\\ &E=-A^2,\quad A_1=1/\eta,\quad A_2=-\mathbf{i}\,/\eta,\quad t>0,\quad \eta>0. \end{aligned}$$

As it is observed from the integrand in (1), near the integration boundaries, the exponential factor is of the form $\exp[\pm\mathbf{i}\,t/(\eta^2\hbar)]$ which exhibits unbounded variation in the limit $\eta \to 0$. The η limit is dealt with by a theorem on the method of stationary phase [12]; actually, the theorem will be extended in Sec. IV. The integration path now goes through a stationary point which depends on physically meaningful magnitudes, and produces finite principal values of the integral (1).

The stationary phase integration is simplified by assuming finite times t which, after scaling $t \to \tau$, obey the condition $0 \le \tau \ll \lambda$. As an example, in the case of an artificial satellite of 1000 kg in an Earth orbit, we listed in [11] the value $\lambda = 4\times 10^{48}$ (in [11] notation was κ instead of λ). In this example, the time intervals allowed by the inequality are much larger than the life time of the solar system. By the bounded time, the integrand is concentrated near the stationary point with the consequence, that, essentially, the integral is given by Hostler's Greens's function, but with dressed entries. E.g., the potential strength parameter Q_0 is replaced by Q_0^{eff} which depends both on time t and the Lambert variables x, y, where $x = r_1 + r_2 + r_{12}$ and $y = r_1 + r_2 - r_{12}$ with $r_{12} = |\mathbf{r_1} - \mathbf{r_2}|$. The approximated propagator inherits from the Green's function the product of the two Whittaker functions $M_{\nu,1/2}(z)\times W_{\nu,1/2}(z')$. For large λ, two of the three entries, namely ν and z or z', both increase linearly with λ. For lack of useful asymptotic formulas in the literature, we will take advantage of a product formula given by Buchholz [1], and determine the constituting integral of the product formula by the saddle-point method; the result appears in terms of elementary functions, exponential and logarithm functions, and by square root factors. In Fig.1 of Sec.VII., we will plot exemplarily real and imaginary parts of the Whittaker product as a function of λ from the built-in formulas of Mathematica [14]. Each of the two curves shows about 30 extrema in the adopted interval $\lambda \in \{15, 36\}$. As it turns out, the corresponding product which is approximated by the saddle-point method, graphically coincides with the primary curves in all details, without visible deviations.

As an exact property, the approximated propagator attains the well known free-particle limit. Due to the restrictive condition on time, the unitary property should be valid only approximately, most likely. If necessary, normalization of an evolving time dependent wave function has to be enforced by an additional normalization factor.

2 Basic definitions

Hostler's Green's function [6], which has to be inserted in the integrand of (1) for G_E, reads

$$G_{\text{Ho}}(\mathbf{r}, \mathbf{r}', E) = \mathbf{i}\, \frac{\Gamma(1 - \mathbf{i}\,\nu)}{4\pi r_{12}} \frac{1}{k} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) W_{\mathbf{i}\,\nu,1/2}(-\mathbf{i}\,kx) M_{\mathbf{i}\,\nu,1/2}(-\mathbf{i}\,ky), \quad (2)$$

where Γ denotes the Gamma function, M, W are the Whittaker functions and (x, y) the Lambert coordinates defined in the Introduction; k is the wave number related to the energy E; together with ν and γ, the definitions are [12]

$$k = \gamma\sqrt{E}, \quad \gamma^2 = 2m/\hbar^2, \quad 2\mathbf{i}\,\nu = \mathbf{i}\,\gamma^2\alpha/k, \quad (3)$$

where $(-\alpha) = G_{grav} m_1 m_2$ is the coupling constant in the Kepler problem, in the Coulomb case, $\alpha = \pm|e_1 e_2|/(4\epsilon_0\pi)$. We will confine ourself to attractive potentials with $\alpha > 0$. The potential-free limit of G_{Ho} comes out as [12]

$$G_{Ho} \to -\exp\left(\mathbf{i}\,kr_{12}\right)/(4\pi r_{12}). \quad (4)$$

The Green's function in (1) has the dimension of its spectral representation, namely $[E^{-1}r^{-3}]$. Its connection to G_{Ho} is given by

$$G_E = -\gamma^2 G_{Ho} \quad \gamma^2 = 2m/\hbar^2. \quad (5)$$

3 Scaling by mean values of the initial wave function

We introduce the following mean values from any initial wave function ψ_0, together with units of time and energy, T_0 and E_0, respectively,

$$R_0 = |\langle\psi_0\, \mathbf{r}\psi_0\rangle|, \quad V_0 = \frac{1}{m}|\langle\psi_0\, \mathbf{p}\psi_0\rangle|, \quad T_0 = \frac{R_0}{V_0}, \quad E_0 = \frac{m}{2}V_0^2. \tag{6}$$

In the macroscopic limit, R_0 is the mean initial distance of the orbiting mass point from the force center, and V_0 the mean initial speed. We mark, at first, dimensionless magnitudes by a tilde as

$$\begin{aligned} A &\equiv \mathbf{i}\sqrt{E} = \sqrt{E_0}\,\tilde{A}, \quad \mathbf{r} = R_0\tilde{\mathbf{r}}, \quad \mathbf{r}' = R_0\tilde{\mathbf{r}}', \quad \lambda = \frac{mR_0V_0}{\hbar}, \\ \gamma^2 &\equiv \frac{2m}{\hbar^2} = \frac{\lambda^2}{R_0^2E_0}, \quad \tau \equiv \frac{t}{2T_0} = \frac{\tilde{t}}{\lambda}, \quad R_0k = \tilde{k} \equiv \mathbf{i}\,\lambda\tilde{A}, \quad K = \frac{\tilde{K}}{R_0^3}, \\ G &= \frac{\tilde{G}}{E_0R_0^3}, \quad A^2\frac{t}{\hbar} = \tilde{A}^2\tilde{t}, \quad T_0 = \frac{R_0}{V_0}, \quad \tilde{E} = \frac{E}{E_0}. \end{aligned} \tag{7}$$

The number λ, classically, refers to the initial angular momentum in units of $\hbar$; it was the main order of magnitude parameter in [11] and [13] (where λ was denoted by κ). The scaled equation (1) now reads with the multiply scaled time $\tilde{t}$

$$\tilde{K}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}', \tilde{t}) = \lim_{\tilde{\eta}\to 0}\left\{\frac{\mathbf{i}}{\pi}\int_{\tilde{A}_1}^{\tilde{A}_2} \mathrm{d}\tilde{A}\,\tilde{A}\exp[\mathbf{i}\,\tilde{A}^2\,\tilde{t}\,]\tilde{G}_{\tilde{E}}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}')\right\}, \quad \tilde{E} = -\tilde{A}^2. \tag{8}$$

In the expression of Hostler's Green's function (2), which has the unit R_0^{-1}, the Lambert coordinates x, y, and r_{12} have dimension R_0. After the substitution $G_{\mathrm{Ho}} \to R_0^{-1}\tilde{G}_{\mathrm{Ho}}$, the scaled $\tilde{G}_{\mathrm{Ho}}$ is given by (2) in terms of the scaled variables, whereby the dimensionless parameter $\mathbf{i}\,\nu$, defined in (3), depends on the integration variable $\tilde{A}$, and has to be replaced by ν_A:

$$\mathbf{i}\,\nu = \nu_A = \mathbf{i}\,\frac{\gamma^2\alpha}{2k}, \quad \nu_A = \frac{\nu_0}{\tilde{A}}, \quad \nu_0 = \frac{\lambda}{2}\,Q_0, \quad Q_0 = \left(\frac{\alpha}{R_0}\right) / \left(\frac{mV_0^2}{2}\right) > 0. \tag{9}$$

Classically, Q_0 is the quotient of the negative potential and kinetic energy at initial time; Q_0 is positive for our assumed positive α; generally, it is of order 1, whereas ν_0 can get large through the factor λ in mesoscopic or macroscopic cases. One obtains the following scaled function

$$\tilde{G}_{\mathrm{Ho}} = -\frac{\Gamma(1-\nu_A)}{4\pi\tilde{r}_{12}}\frac{1}{\mathbf{i}\,\tilde{k}}\left(-\frac{\partial}{\partial\tilde{y}} + \frac{\partial}{\partial\tilde{x}}\right)W_{\nu_A,1/2}(-\mathbf{i}\,\tilde{k}\tilde{x})M_{\nu_A,1/2}(-\mathbf{i}\,\tilde{k}\tilde{y}). \tag{10}$$

According to (5), we make use of $G_E = -\gamma^2 G_{\rm Ho}$ which implies

$$(E_0 R_0^3)^{-1}\tilde{G}_E = -\gamma^2 (R_0)^{-1}\tilde{G}_{\rm Ho}, \quad \text{or} \quad \tilde{G}_E = -\lambda^2 \tilde{G}_{\rm Ho}. \tag{11}$$

In (8), we replace $\tilde{G}_E$ by $-\lambda^2\tilde{G}_{\rm Ho}$, and from (7) we set in (2) $(-\mathbf{i}\,\tilde{k}) = \lambda\tilde{A}$; moreover, we denote $\tilde{t} \equiv c$ and will elsewhere omit the tilde marks to write the propagator as follows

$$K(\mathbf{r},\mathbf{r}',c) = \mathbf{i}\,\frac{\lambda}{4\pi^2\, r_{12}} \lim_{\eta\to 0}\left\{\int_{A_1}^{A_2} \mathrm{d}A\, \exp[\mathbf{i}\,A^2\,c]\Gamma(1-\nu_A)\times \right. \tag{12}$$

$$\left.\left(-\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\right) P_{\nu_A,1/2}(A,x,y)\right\}, \quad P_{\nu_A,1/2} = W_{\nu_A,1/2}(\lambda A\, x) M_{\nu_A,1/2}(\lambda A\, y),$$

where in terms of the original time t one has $c = \lambda\, t/(2T_0)$ with $\eta = 0$.

The scaled propagator K is tested by the potential-free limit $\nu_A \to 0$ which according to [3] reads

$$K^{(0)} = \exp(-3\pi\,\mathbf{i}\,/4)\left(\frac{m}{2\pi\hbar t}\right)^{3/2} \exp\left[\frac{\mathbf{i}\, m(\mathbf{r}-\mathbf{r}')^2}{2\hbar t}\right], \quad t \geq 0. \tag{13}$$

The above $K^{(0)}$ has dimension R_0^{-3}; it is consistent with the path integral result obtained for the 1-dimensional configuration space by Eq.(3.3) in [2], which can easily be extended to three dimensions. Details of deriving (13) from (12) are given in Subsec.1. of Appendix B.

4 Integration with the method of stationary phase

We apply a theorem on the method of stationary phase. It is based on the following lemma, which was proved in [12] for non-negative integers n; in Appendix A the lemma is extended to negative integers. One defines the integrals

$$B_n = \lim_{\eta\to 0}\int_{1/\eta}^{-\mathbf{i}\,/\eta} \mathrm{d}A\, A^n \exp[\mathbf{i}\, cA^2 - s\,A], \quad \eta > 0, \quad n \in \mathbf{Z}, \tag{14}$$

adds a small negative imaginary part to c, and formulates the
Lemma

$$\begin{aligned} B_n &= (-\mathbf{i}\,)\exp[-\mathbf{i}\, cA_s^2]\int_{-\infty}^{\infty} \mathrm{d}a\,(A_s - \mathbf{i}\,a)^n \exp[-\mathbf{i}\,c\,a^2], \quad n \in \mathbf{Z}, \qquad (15)\\ A_s &= \frac{-\mathbf{i}\,s}{2c}, \quad s = \lambda\frac{x-y}{2} > 0, \quad c = \lambda\,\tau - \mathbf{i}\,\eta^{3/2}, \quad \tau = \frac{t}{2T_0} > 0, \end{aligned}$$

where A_s is the stationary point of the exponent of the integrand in (14). It should be noticed that the particular form of the small imaginary part of c is useful in the proof of the stationary phase theorem, see Appendix A and [12]. With the aid of the lemma, one immediately infers the

Theorem

If the amplitude $f(A)$ is analytic in the half plane $\mathrm{Re}(A) > 0$, then

$$\lim_{\eta\to 0}\int_{1/\eta}^{-\mathbf{i}/\eta} \mathrm{d}A\, f(A)\exp[\mathbf{i}\, cA^2 - s\, A] = \tag{16}$$

$$(-\mathbf{i}\,)\exp[-\mathbf{i}\, cA_s^2]\int_{-\infty}^{\infty} \mathrm{d}a\, f\,(A_s - \mathbf{i}\, a)\exp[-\mathbf{i}\, c\, a^2]. \tag{17}$$

By the theorem, the integration interval of the highly oscillating integrand of (16) is concentrated around the physically meaningful stationary point A_s.

Let us apply the theorem to the Fourier transformation of the propagator (12). In the exponent of the integral (16), we need the linear term $(-sA)$; so we insert the unit $1 \equiv \exp[-sA] \times \exp[sA]$. Furthermore, we introduce the abbreviation γ_K by

$$\gamma_K = -\frac{\lambda}{4\pi^2\, r_{12}}\exp\left(-\mathbf{i}\, cA_s^2\right) \tag{18}$$

to write

$$\begin{aligned} K(\mathbf{r},\mathbf{r}',c) &= \gamma_K\int_{-\infty}^{\infty}\mathrm{d}a\ \exp(-\mathbf{i}\, c\, a^2) f_K\,(A)\,, \quad A = A_s - \mathbf{i}\, a, \\ f_K(A) &= \exp[s\, A]\,\Gamma(1-\nu_A)\left(-\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\right)P_{\nu_A,1/2}(A,\, x,\, y). \end{aligned} \tag{19}$$

In Subsec.2. of Appendix B, we show that the free particle limit of (19), with $\nu_A = 0$, restores $(R_0^3 K^{(0)})$ provided that one chooses

$$s = \lambda(x-y)/2 = \lambda\, r_{12}. \tag{20}$$

As is implied by (B5), after scaling, the free particle propagator $K^{(0)}$ can also be written as

$$R_0^3 K^{(0)} = \frac{\lambda^2 A_s \exp\left(-\mathbf{i}\, c\, A_s^2\right)}{4\pi^{3/2} r_{12}\sqrt{\mathbf{i}\, c}} = (-1)^{5/4}\lambda^3\exp\left(\mathbf{i}\,\frac{s^2}{4c}\right)(4\pi\, c)^{-3/2} \tag{21}$$

which allows for writing the pre-factor of the propagator in (19) as follows

$$\gamma_K = \gamma_0\,\left(R_0^3 K^{(0)}(\mathbf{r},\mathbf{r}',c)\right), \quad \gamma_0 = -\frac{1}{\lambda\, A_s}\sqrt{\frac{\mathbf{i}\, c}{\pi}}. \tag{22}$$

5 Restriction to a bounded time interval

Due to the factor $\exp(-\mathbf{i}\, c\, a^2)$, the effective integration interval in (19) is confined to a finite range $-b < a < b$, where b is of order $1/\sqrt{c}$. When the function $f_K(A) \equiv f_K\left(A_s(1+a_s)\right)$ is Taylor expanded with respect to $a_s = -\mathbf{i}\, a/A_s$, it is seen that, by parity, only even powers in a_s survive the a integration. So, if we approximate $f_K(A)$ by $f_K(A_s)\,(1+E_{err}))$, the error term is expected to be of order

$$E_{err} = \mathcal{O}\left(|\frac{1}{A_s^2 c}|\right) = \mathcal{O}\left(\frac{4}{r_{12}^2}\,\frac{\tau}{\lambda}\right), \quad \tau = \frac{t}{2T_0}, \quad \lambda = \frac{mR_0V_0}{\hbar}, \tag{23}$$

which implies that $E_{err} \ll 1$, if the scaled time τ is small compared with λ, provided that the distance r_{12} stays away from zero:

$$\tau \ll (1/4) r_{12}^2\, \lambda. \tag{24}$$

In Appendix C, we give examples which support the estimate for E_{err} in (23).

In Tab.1 of [11], we listed two-body examples with the corresponding λ values (there denoted by κ). For an electron - proton scenario we wrote the number $\lambda = 4.5$, for a proton-proton case $\lambda = 139$, for an artificial satellite of 10^3 kg in an Earth orbit we stated $\lambda = 4 \times 10^{48}$. Thus, in mesoscopic und macroscopic examples, the condition (24) generally allows for large time intervals $\tau = t/(2T_0)$, even beyond physically reasonable limits (e.g. life time of the solar system).

In conclusion, we assume that we obtain a reasonable approximation of the propagator when we replace in the integrand (19) $f_K(A)$ by $f_K(A_s)$. For the remaining a integral, we take into account the small negative imaginary part of c

$$c = \lambda\tau - \mathbf{i}\,\eta^{3/2}, \quad A_s = -\mathbf{i}\,\lambda\,(r_{12}/2)\left(\lambda\tau - \mathbf{i}\,\eta^{3/2}\right)^{-1}, \quad \eta > 0 \tag{25}$$

to obtain

$$a_0 = \int_{-\infty}^{\infty} \mathrm{d}a\, \exp[-\mathbf{i}\, c\, a^2) = \sqrt{-\mathbf{i}\,\pi/c}, \tag{26}$$

and so we propose the approximation $K \to K_\lambda$ with

$$\begin{aligned} K_\lambda(\mathbf{r},\mathbf{r}',c) \;&=\; \gamma_K a_0 f_K\left(A_s\right), \quad f_K(A_s) = \exp[s\, A_s]\,\Gamma(1-\nu_{A_s}) \times \\ &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) P_{\nu_{A_s},1/2}\left(A_s, x, y\right). \end{aligned} \tag{27}$$

For the potential-free test, with $Q_0 = \nu_A = 0$, we use (22) for γ_K, (26) for a_0, and from (B1) the relation

$$W_{0,1/2}(\lambda A_s\, x) M_{0,1/2}(\lambda A_s\, y) = 2\exp[-\lambda\, A_s x/2]\sinh[\lambda\, A_s y/2], \tag{28}$$

and immediately find the exact potential-free property (21), namely

$$\lim_{Q_0\to 0} K_\lambda = R_0^3\, K^{(0)}(\mathbf{r},\mathbf{r}',c). \tag{29}$$

6 Asymptotics for large λ

The approximated propagator K_λ has the structure of the Green's function (2). But, one now has "dressed" magnitudes: the first entry of the Whittaker functions in (2), $\nu_{A_s} = \lambda(Q_0/2)/A_s$, which is related to the potential strength parameter Q_0, is replaced by

$$\nu_{A_s} = \lambda Q_0^{(\mathrm{eff})}, \quad Q_0^{(\mathrm{eff})} = \mathbf{i}\,\frac{4\tau}{x-y}\,Q_0, \quad \tau = \frac{t}{2T_0}; \tag{30}$$

in the third entries of the Whittaker functions, the Lambert coordinates x and y, respectively, are dressed by the factor $A_s = -\mathbf{i}\,(x-y)/(4\tau)$:

$$k\,x \to \lambda X^{(\mathrm{eff})},\ X^{(\mathrm{eff})} = A_s x,\ k\,y \to \lambda Y^{(\mathrm{eff})},\ Y^{\mathrm{eff}} = A_s y. \tag{31}$$

It should be noticed that both the first and the third entry of the Whittaker functions increase linearly with λ. For this case, we could not find useful asymptotic formulas in the literature.

However, we will succeed with the aid of a product formula by Buchholz, see p. 86 of [1]. One defines the product

$$P_{\nu_A,\mu/2}(z) = W_{\nu_A,\mu/2}(a_1 z) M_{\nu_A,\mu/2}(a_2 z) \tag{32}$$

to write the Buchholz formula as follows

$$\begin{aligned} P_{\nu_A\mu/2}(z) \quad &= \quad \frac{z\sqrt{a_1 a_2}\,\Gamma(1+\mu)}{\Gamma[(1+\mu)/2 - \nu_A]} \times \\ &\int_0^\infty \mathrm{d}v\ \exp\left[-z\cosh(v)\frac{a_1+a_2}{2}\right] \frac{I_\mu(z\sqrt{xy}\,\sinh(v))}{\tanh^{2\nu_A}(v/2)}. \end{aligned} \tag{33}$$

The integral exists under the conditions

$$\mathrm{Re}[(1+\mu)/2 - \nu_A] > 0, \quad \mathrm{Re}(\mu) > 0, \quad \text{and} \quad a_1 > a_2. \tag{34}$$

As compared to the original equation in [1], we replaced the regularized Whittaker function $\mathcal{M}$ by the common function M, according to Eq.(7) in §2 of [1] $\mathcal{M}_{\nu_A,\mu/2} = M_{\nu_A,\mu/2}/\Gamma(1+\mu)$. In view of the approximated propagator K_λ defined in (27), we identify parameters as follows

$$\nu_{A_s} = \frac{\nu_0}{A_s}, \quad \nu_0 = \frac{\lambda}{2} Q_0, \quad \mu = 1, \quad z = \lambda A_s, \quad a_1 = x, \quad a_2 = y, \tag{35}$$

and write

$$P_{\nu_{A_s},1/2} = \frac{\lambda A_s \sqrt{xy}}{\Gamma(1-\nu_{A_s})} \int_0^\infty \mathrm{d}v\ \exp\left[-\lambda A_s \cosh(v)\frac{x+y}{2}\right] \frac{I_1(\lambda A_s\sqrt{xy}\,\sinh(v))}{\tanh^{2\nu_{A_s}}(v/2)}. \tag{36}$$

As to the conditions (34), the last one implies $x > y$ which is fulfilled if $\mathbf{r_1} \neq \mathbf{r_2}$, the condition $\mathrm{Re}(\mu) > 0$ is obvious, and $\mathrm{Re}(1-\nu_{A_s}) > 0$ holds true since ν_{A_s} is purely imaginary in the limit $\eta \to 0$.

As is needed for the propagator, we apply the differential operator $(\partial_x - \partial_y)$. After some simplifications and the substitution $v \to u = \cosh(v)$, the modified first order Bessel function I_1 is replaced by the zero order Bessel function J_0 as follows

$$(\partial_x - \partial_y) P_{\nu_A,1/2} = d_A L_A, \quad d_A = -\frac{\lambda^2 A^2 (x-y)}{2\Gamma(1-\nu_A)}, \quad k = \mathbf{i}\,\lambda\, A, \tag{37}$$

$$L_A \;=\; \int_1^\infty \mathrm{d}u\, \exp[-\lambda A u(x+y)/2] J_0\left(\mathbf{i}\,\lambda A\sqrt{xy}\sqrt{u^2-1}\right)\left(\frac{u+1}{u-1}\right)^{\nu_A}. \tag{38}$$

By (12) and (38), the scaled, exact propagator $K(\mathbf{r}, \mathbf{r}', t)$ can be written as

$$K(\mathbf{r}, \mathbf{r}', c) = \mathbf{i}\,\frac{\lambda^3}{4\pi^2} \lim_{\eta\to 0} \int_{A_1}^{A_2} \mathrm{d}A\, \exp[\mathbf{i}\, c\, A^2] A^2\, L_A, \tag{39}$$

where we made use of $x - y = 2r_{12}$; the factor $\Gamma(1-\nu_A)$ has dropped out. As is shown in Subsec.2. of Appendix B, the potential-free limit of the above K, reproduces $K^{(0)}$ as stated in (13), after multiplication by R_0^3.

For the approximated propagator K_λ, given in (27), we need the function L_A of (38) at the stationary point $A = A_s$. In Appendix D, the corresponding integral is asymptotically calculated for large λ by the saddle-point method. To represent the result, we introduce the following abbreviations

$$\xi \;=\; 32\,Q_0 \frac{\tau^2}{(x-y)^2}, \quad \Xi = \mathbf{i}\,\lambda \frac{(x-y)\xi}{16\,\tau}, \quad W_{u,v} = \sqrt{u(v+\xi)}, \tag{40}$$

$$\Xi_1 \;=\; \frac{-x+y-W_{x,x}+W_{y,y}}{x-y-W_{x,x}+W_{y,y}}, \quad \Xi_2 = \frac{x-y+W_{x,x}+W_{y,y}}{-x+y+W_{x,x}+W_{y,y}},$$

where one can show that $\Xi_{1,2} > 1$, if $x > y$; , Ξ is purely imaginary. The result of the saddle-point integration (abbreviated by sp) is now written as follows

$$L_{A_s}^{sp} \;=\; \beta\, L^{sp}, \quad L^{sp} = (\Lambda_1 + \mathbf{i}\,\Lambda_2)\,(W_{x,x}W_{y,y})^{-1/2}, \quad \beta = \frac{4\tau}{\lambda(x-y)^2},$$

$$\Lambda_1 = \exp\left[\mathbf{i}\,\lambda \frac{(x-y)(W_{x,x}+W_{y,y})}{8\tau}\right](W_{x,y} - W_{y,x})\,(\Xi_1)^{\Xi},$$

$$\Lambda_2 = \exp\left[\mathbf{i}\,\lambda \frac{(x-y)(W_{x,x}-W_{y,y})}{8\tau}\right](W_{x,y} + W_{y,x})\,(\Xi_2)^{\Xi}. \tag{41}$$

In the potential-free limit, which amounts to $\xi \to 0$, we find

$$\lim_{\xi\to 0} L_{A_s}^{sp} = \frac{\exp\left(-\lambda A_s r_{12}\right)}{\lambda A_s r_{12}} \quad \lim_{\xi\to 0} L^{sp} = \frac{1}{\beta} \lim_{\xi\to 0} L_{A_s}^{sp}, \quad A_s = -\mathbf{i}\,\frac{r_{12}}{2\,\tau}, \tag{42}$$

which according to (B14) agrees with the corresponding limit, $L_{A_s}^{(0)}$, of the exact integral L_{A_s}. Eventually, we approximate in (27) the factor $F_K(A_s)$ in terms of the saddle-point calculation. With the aid of (27), (38), and (41), we get

$$F_K(A_s) \to \frac{\lambda}{8\tau}(x-y)\exp\left(-\mathbf{i}\,\frac{(x-y)^2\lambda}{8\tau}\right)L^{sp}, \tag{43}$$

which, by (27), leads to

$$K_\lambda \to K_\lambda^{sp} = -\mathbf{i}\,\frac{\left(R_0^3K^{(0)}\right)}{2}\exp\left(-\mathbf{i}\,\frac{(x-y)^2\lambda}{8\tau}\right)L^{sp}. \tag{44}$$

In view of (42), the potential-free limit of $K_\lambda^{sp} \to \left(R_0^3K^{(0)}\right)$ comes out correctly.

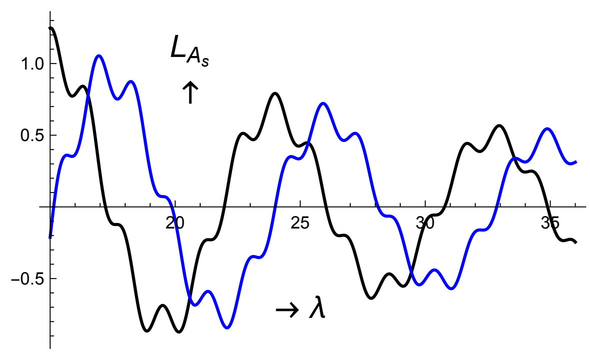

Figure 1: Plots of the real (black curve) and imaginary (blue curve) parts of the function L_{A_s} which is defined by Eq.(45) in terms of the Whittaker functions. The Lambert coordinates are chosen by $x = 2$ and $y = 1$; time parameter is $\tau \equiv t/(2T_0) = 2$, and the potential strength parameter $Q_0 = 3/2$.

7 A numerical check of the saddle-point integration

By (37), the function L_A is expressed in terms of the product of Whittaker functions:

$$L_A = \frac{1}{d_A}(\partial_x - \partial_y)P_{\nu_A,1/2} \equiv \frac{1}{d_A}(\partial_x - \partial_y)W_{\nu_A,1/2}(\lambda A\,x)M_{\nu_A,1/2}(\lambda A\,y), \tag{45}$$

where A has to be replaced by A_s. By Mathematica [14], the differentiations are rendered analytically in terms of Whittaker functions, partially, with shifted first entry. We take advantage of the built-in functions WhittakerM[...] and WhittakerW[...], and choose parameters as follows: $x = 2$,

$y = 1$, $\tau = 2$, $\eta = 0$, and $Q_0 = 3/2$. Then, with $\nu_{A_s} = 2\mathbf{i}\,\lambda\tau Q_0/(x-y)$, the function L_{A_s} depends on λ only. In Fig.1, the real and imaginary parts of L_{A_s} are plotted in the interval $15 \le \lambda \le 36$. For larger λ values, numerical instabilities emerge with standard setting of Mathematica precision.

How do the two curves of Fig.1 compare with those obtained from $L^{sp}_{A_s}$, given in (41), derived by the saddle-point method? A graphical representation is redundant, since the corresponding curves graphically coincide with those of Fig.1 in all details, without visible deviations. The computing time for rendering Fig.1 took about 20 seconds, whereas the graphical representation from (41) is faster by about a factor thousand. In Tab.1, we list root-mean-square deviations (RMSD) in different λ intervals.

RMSD of L_A curves

$\lambda\ \in$	{3,8}	{8,13}	{13,18}	{18,23}	{23,28}	{28,33}
Re(L_{A_s})	0.0192	0.00376	0.0015	0.00077	0.00050	0.00037
Im(L_{A_s})	0.0170	0.0034	0.00142	0.00082	0.00057	0. 00040

Table 1: Root-mean-square deviations (RMSD) of L_{A_s} curves calculated in two different ways: directly from the product of Whittaker functions, and from the Buchholz formula [1], whose integral is calculated by the saddle-point method.

8 Summary

As compared to a previous result [12], the theorem on the method of stationary phase is extended to include negative powers of A in the integrand. The theorem is applied to the representation of the Coulomb-Kepler propagator by means of a, suitably modified, Fourier transformation of the exactly available Green's function [6]. By the theorem, the Fourier integral is determined by a path, which goes through a stationary point A_s. The latter is a function of the two Lambert coordinates x, y and of time t. The stationary phase integral leads to an elementary compact form K_λ of the propagator, provided that the time interval is restricted by a condition which is physically reasonable in mesoscopic or macroscopic cases. K_λ has the structure of the Green's function, and thus, essentially, is the product of the two Whittaker functions, but depending on "dressed" coupling constant and Lambert variables. Asymptotically, for large λ, the Whittaker product is brought into an elementary form by means of a saddle-point integration. The approximated propagator is in a compact analytical form and should be suitable for docking to any initial wave function.

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A Proof of stationary phase lemma for negative powers

In [12], the following lemma was proved for non-negative integer powers A^n:
Lemma

$$j_n = \lim_{\eta\to 0}\int_{1/\eta}^{-\mathbf{i}/\eta} \mathrm{d}A\, A^n \exp[\mathbf{i}\, cA^2 - s\, A] \quad \text{(A1)}$$

$$= (-\mathbf{i}\,)\exp[-\mathbf{i}\, cA_s^2]\int_{-\infty}^{\infty} \mathrm{d}a\,(A_s - \mathbf{i}\, a)^n \exp[-\mathbf{i}\, c\, a^2], \quad n = 0, 1, 2, .. \quad \text{(A2)}$$

where A_s is the stationary point of the exponent in (A1) with

$$A_s = -\mathbf{i}\, s/(2c), \quad s = \lambda\, r_{12}, \quad c = \lambda\tau - \mathbf{i}\,\eta^{3/2}, \quad \tau = t/(2T_0) > 0, \ \eta > 0. \quad \text{(A3)}$$

We prove that the lemma is also true for negative integers n. To this end, we set r=(-n) and define the integrals

$$J_r = \lim_{\eta\to 0}\int_{1/\eta}^{-\mathbf{i}/\eta} \mathrm{d}A\, \frac{1}{A^r} \exp[\mathbf{i}\, cA^2 - s\, A], \quad r = 0, 1, 2, ... \quad \text{(A4)}$$

$$R_r = \lim_{\eta\to 0}\left[(-\mathbf{i}\,)\exp[-\mathbf{i}\, cA_s^2]\int_{-\infty}^{\infty} \mathrm{d}a\, \frac{1}{(A_s - \mathbf{i}\, a)^r} \exp[-\mathbf{i}\, c\, a^2]\right]. \quad \text{(A5)}$$

By means of partial integration, one gets the following difference equations which are the same, up to possibly different initial values $(J_0,\ J_1)$ and $(R_0,\ R_1)$; the free terms vanish in view of $c = \lambda\tau - \mathbf{i}\,\eta^{3/2}$ and $s > 0$:

$$J_r = \frac{2\mathbf{i}\, c}{r-1}\left[J_{r-2} - A_s J_{r-1}\right], \quad r = 2, 3, \ldots \quad \text{(A6)}$$

$$R_r = \frac{2\mathbf{i}\, c}{r-1}\left[R_{r-2} - A_s R_{r-1}\right], \quad r = 2, 3, \ldots \quad \text{(A7)}$$

It remains to show that $J_0 = R_0$ and $J_1 = R_1$. Then, recursively $J_r = R_r$ for $r = 2, 3, ...$, and the Lemma is extended to negative integers n.

From Eq.(50) in [12], we have

$$R_0(s) \equiv J_0(s) = (-1)^{5/4}\sqrt{\frac{\pi}{c}}\, \exp\left(-\mathbf{i}\, cA_s^2\right) = (-1)^{5/4}\sqrt{\frac{\pi}{c}}\, \exp\left(\mathbf{i}\, \frac{s^2}{4c}\right). \quad \text{(A8)}$$

For $r = 1$, we use the auxiliary integral transforms

$$\frac{1}{A} = \int_0^\infty \mathrm{d}u \, \exp[-A\,u], \quad \mathrm{Re}[A] > 0 \quad \text{and} \tag{A9}$$

$$\frac{1}{A_s - \mathbf{i}\,a} = \int_0^\infty \mathrm{d}u \, \exp[-(A_s - \mathbf{i}\,a)\,u], \quad \mathrm{Re}[A_s - \mathbf{i}\,a] = \frac{s\,\eta^{3/2}}{c\,c^*} > 0, \tag{A10}$$

and obtain

$$J_1 = \int_0^\infty \mathrm{d}u \lim_{\eta\to 0} \int_{1/\eta}^{-\mathbf{i}/\eta} \mathrm{d}A \, \exp[\mathbf{i}\,cA^2 - (s+u)\,A] = \int_0^\infty \mathrm{d}u \, J_0(s+u)$$

$$= -\mathbf{i}\,\pi \mathrm{Erfc}[A_s\sqrt{\mathbf{i}\,c}]; \tag{A11}$$

$$R_1 = -\mathbf{i}\, \exp[-\mathbf{i}\,cA_s^2]\,r_1, \;\; r_1 = \int_{-\infty}^\infty \mathrm{d}a \, \frac{\exp[-\mathbf{i}\,c\,a^2]}{A_s - \mathbf{i}\,a} = \tag{A12}$$

$$\int_0^\infty \mathrm{d}u \, \exp[-A_s u]\,\rho_1(u), \;\; \rho_1(u) = \int_{-\infty}^\infty \mathrm{d}a \, \exp[-\mathbf{i}\,c\,a^2 + \mathbf{i}\,u\,a] = \tag{A13}$$

$$\sqrt{\frac{\pi}{\mathbf{i}\,c}} \exp\left[\mathbf{i}\,\frac{u^2}{4c}\right], \quad r_1 = \pi \exp[\mathbf{i}\,c\,A_s^2]\,\mathrm{Erfc}[A_s\sqrt{\mathbf{i}\,c}],$$

which gives rise to the desired result

$$R_1 = -\mathbf{i}\,\pi\,\mathrm{Erfc}\left[A_s\sqrt{\mathbf{i}\,c}\right] \equiv J_1. \tag{A14}$$

B Tests of potential-free case

B.1 Test of expression (12) of the propagator K

Zero coupling with $Q_0 = \nu_A = 0$ implies elementary Whittaker functions as follows

$$M_{0,1/2}(z) = 2\sinh(z/2), \quad W_{0,1/2}(z) = \exp(-z/2). \tag{B1}$$

Using the explicit Lambert variables defined in the Introduction, we obtain

$$\left(-\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) P_{\nu_A,1/2}(A,x,y) = -\lambda\,A \exp[-s\,A], \quad s = \lambda\,(x-y)/2. \tag{B2}$$

From (12) and $\Gamma(1) = 1$, one obtains, introducing the notation $K_0 = \lim_{\nu_A\to 0} K$,

$$K_0 = +\mathbf{i}\,\frac{\lambda^2}{4\pi^2\,r_{12}}(-\partial_s B_0), \quad B_0 = \lim_{\eta\to 0}\left\{\int_{1/\eta}^{-\mathbf{i}/\eta} \mathrm{d}A \, \exp[\mathbf{i}\,A^2\,c\, - sA]\right\}. \tag{B3}$$

$$B_0 = (-\mathbf{i}\,)\exp(-\mathbf{i}\,c\,A_s^2)\int_{-\infty}^\infty \mathrm{d}a \exp(-\mathbf{i}\,c\,a^2) = (-1)^{5/4}\sqrt{\frac{\pi}{c}}\exp\left(\mathbf{i}\,\frac{s^2}{4c}\right). \tag{B4}$$

After inserting B_0 into K_0, given in (B3), we find

$$K_0 = (-1)^{5/4}\lambda^3 \exp\left(\mathbf{i}\,\frac{s^2}{4c}\right)(4\pi\, c)^{-3/2}. \tag{B5}$$

We remind that K_0 is dimensionless and depends on the scaled variables. In order to arrive at the textbook formula (13), we have to invert the substitutions (7) as

$$K_0 = R_0^3 K^{(0)},\ s \to \lambda r_{12}/R_0,\ r_{12} \to r_{12}/R_0,\ c \to (\lambda/2)(R_0/V_0)t. \tag{B6}$$

Eventually, we use $\lambda = mR_0V_0/\hbar$ to arrive at the desired result (13).

B.2 Test of expression (19) of the propagator K

With the aid of (B1) and (B2), after setting $\nu_A = 0$ and $A = A_s - \mathbf{i}\,a$, one obtains

$$f_K(A_s) = \exp[s\,A_s]\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) P_{0,\frac{1}{2}}(A_s,\, x,\, y) = -\lambda\,(A_s - \mathbf{i}\,a). \tag{B7}$$

With the aid (22) for γ_K and (26) for a_0, one immediately gets

$$K_0 = -\gamma_K \lambda\, A_s a_0 = R_0^3 K^{(0)}. \tag{B8}$$

B.3 Test of expression (39) of the Propagator K

The potential-free case, with $\nu_A = 0$, leads to the integral $L_A \to L_A^{(0)}$, which can be found from formula 6.645 2. in [5] as

$$L_A^{(0)} = \int_1^\infty \mathrm{d}u\, \exp[-\lambda A u(x+y)/2] J_0\left(\mathbf{i}\,\lambda A\sqrt{xy}\sqrt{u^2-1}\right) \tag{B9}$$

$$= \sqrt{2/\pi}\left(\alpha_1^2 + \beta_1^2\right)^{-1/4} K_{1/2}(\sqrt{\alpha_1^2+\beta_1^2}), \tag{B10}$$

$$\alpha_1 = \lambda A(x+y)/2, \quad \beta_1 = \mathbf{i}\,\lambda A\sqrt{xy},$$

where $K_{1/2}$ is a modified Bessel function of the second kind; it is elementary. We get, taking the plus sign of the square root,

$$Z = \sqrt{\alpha_1^2+\beta_1^2} = \lambda A(x-y)/2, \tag{B11}$$

$$K_{1/2}(Z) = \sqrt{\pi/(2Z)}\exp[-Z] = \sqrt{\frac{\pi}{2}}\,\frac{\exp(-A\lambda r_{12})}{\sqrt{A\lambda r_{12}}}, \quad r_{12} = (x-y)/2, \tag{B12}$$

and, thus,

$$L_A^{(0)} = \frac{1}{\lambda\, A\, r_{12}}\exp(-\lambda A\, r_{12}). \tag{B13}$$

According to (39), we are left with the following A integral

$$K_0(\mathbf{r},\mathbf{r}',t) \;=\; \mathbf{i}\,\frac{\lambda^2}{4\pi^2 r_{12}}\,B_1, \quad s=\lambda r_{12}, \quad c=\lambda\,\tau-\mathbf{i}\,\eta^{3/2}, \tag{B14}$$

$$B_1 \;=\; \lim_{\eta\to 0}\int_{A_1}^{A_2} \mathrm{d}A\,A\,\exp[\mathbf{i}\,A^2\,c\,-\,sA]. \tag{B15}$$

The integral B_1 is calculated by the theorem of stationary phase (16), (17).

$$B_1 = -\frac{s}{2c}\sqrt{\frac{\pi}{\mathbf{i}\,c}}\,\exp\left(\mathbf{i}\,\frac{s^2}{4c}\right). \tag{B16}$$

To recover the propagator $K^{(0)}$, see (13), from (B14) and (B16), we invert the scaling according to (7) and use the standard principal value $(-1)^{5/4}\exp[\mathbf{i}\,3\pi/4] = 1$.

C Examples for the error term in (23)

In the theorem (16) and (17), we choose four special amplitudes $f(A)$:

$$f_1=A^3, \quad f_2=A^5, \quad f_3=\frac{1}{A}, \quad f_4=\frac{1}{A^3}, \quad A=A_s-\mathbf{i}\,a, \tag{C1}$$

and define the integrals

$$F_j \;=\; (-\mathbf{i}\,)\exp(-\mathbf{i}\,cA_s^2)\int_{-\infty}^{\infty}\mathrm{d}a\,\exp(-\mathbf{i}\,c\,a^2)\,f_j(A_s-\mathbf{i}\,a), \tag{C2}$$

$$F_j^{(0)} \;=\; (-\mathbf{i}\,)\exp(-\mathbf{i}\,cA_s^2)\,f_j(A_s)\int_{-\infty}^{\infty}\mathrm{d}a\,\exp(-\mathbf{i}\,c\,a^2), \quad j=1,2,3,4, \tag{C3}$$

where c has the small negative imaginary part $(-\mathbf{i}\,\eta^{3/2})$. The quotients $q_j := F_j/F_j^{(0)}$ should tend to 1 in the limit $\tau/\lambda\to 0$. In the first two cases, one rigorously obtains

$$q_1 = 1-\mathbf{i}\,\frac{6}{r_{12}^2}\,\frac{\tau}{\lambda}, \quad q_2 = 1-\mathbf{i}\,\frac{20}{r_{12}^2}\frac{\tau}{\lambda}-\frac{60}{r_{12}^4}\left(\frac{\tau}{\lambda}\right)^2. \tag{C4}$$

As to q_3 and q_4, we use the recurrence equation (A7) which leads to

$$F_4 \;\equiv\; R_3 = c\left(c\,A_s R_0 + \mathbf{i}\,R_1 - 2c\,A_s^2 R_1\right). \tag{C5}$$

From (A8) and (A14), we have the explicit expressions

$$R_0 \;=\; (-1)^{5/4}\sqrt{\frac{\pi}{c}}\,\exp\left(-\mathbf{i}\,cA_s^2\right) \text{ and } F_3\equiv R_1 = -\mathbf{i}\,\pi\,\mathrm{Erfc}\left[A_s\sqrt{\mathbf{i}\,c}\right]. \tag{C6}$$

Furthermore, we make use of

$$F_3^{(0)} = -\mathbf{i}\, a_0 \exp\left(-\mathbf{i}\, c A_s^2\right) \frac{1}{A_s}, \quad F_4^{(0)} = -\mathbf{i}\, a_0 \exp\left(-\mathbf{i}\, c A_s^2\right) \frac{1}{A_s^3}, \tag{C7}$$

where $a_0 = \sqrt{-\mathbf{i}\,\pi/c}$. In the argument of Erfc, the factor A_s does not depend on λ, which implies that $z = A_s\sqrt{\mathbf{i}\,c}$ is proportional to $\sqrt{\lambda}$. So, we approximate Erfc(z) asymptotically for large λ as

$$\mathrm{Erfc}(z) \to \frac{1}{z\sqrt{\pi}} \exp(-z^2) \left[1 - \frac{1}{2z^2} + \frac{3}{4z^4} - \frac{15}{8z^6} + \mathcal{O}(z^{-8})\right]. \tag{C8}$$

Eventually, we obtain

$$q_3 \to 1 - \mathbf{i}\, \frac{2}{r_{12}^2} \frac{\tau}{\lambda} + \mathcal{O}\left(\frac{\tau}{r_{12}^2 \lambda}\right)^2, \quad q_4 \to 1 - \mathbf{i}\, \frac{12}{r_{12}^2} \frac{\tau}{\lambda} + \mathcal{O}\left(\frac{\tau}{r_{12}^2 \lambda}\right)^2. \tag{C9}$$

D Application of the saddle-point method

The integral for L_A defined in (38) is calculated by the saddle-point method for large λ. To this end, we start from the asymptotic approximation of the Bessel function $J_0(z)$ for large $|z|$:

$$\begin{aligned} J_0(z) &= \frac{1}{\sqrt{2\pi z}} \left[\exp\left(\mathbf{i}\,(\pi/4 - z)\right) + \exp\left(-\mathbf{i}\,(\pi/4 - z)\right)\right] \left\{1 + \mathcal{O}(z^{-1/2})\right\}, \\ & z \equiv z(u) = \mathbf{i}\,\lambda A \sqrt{x\,y}\, \sqrt{u^2 - 1}, \quad u \geq 1. \end{aligned} \tag{D1}$$

With

$$L_A = \int_1^\infty \mathrm{d}u\, l(u), \tag{D2}$$

$$l(u) = \exp[-\lambda A u(x+y)/2] J_0\left(\mathbf{i}\,\lambda A \sqrt{xy(u^2-1)}\right) \left(\frac{u+1}{u-1}\right)^{\nu_A}, \tag{D3}$$

we approximate the integrand $l(u)$ as

$$l(u) \to l_{asy}(u) = \frac{1}{\sqrt{2\pi z}} \left\{\exp(Z_1) + \exp(Z_2)\right\}, \tag{D4}$$

$$\begin{aligned} Z_1 &= -\lambda A u(x+y)/2 + \mathbf{i}\,(z - \pi/4) + \nu_A \left[\ln(u+1) - \ln(u-1)\right], \\ Z_2 &= -\lambda A u(x+y)/2 - \mathbf{i}\,(z - \pi/4) + \nu_A \left[\ln(u+1) - \ln(u-1)\right]. \end{aligned} \tag{D5}$$

We need L_A at the value $A = A_s = -\mathbf{i}\,(x-y)/(4\tau)$, where $\nu_{A_s} = 2\mathbf{i}\,\lambda\, Q_0\tau/(x-y)$, which implies that both exponents $Z_{1,2}$ increase linearly with λ. So, for large λ, we Taylor expand $Z_{1,2}(u)$ at the stationary points U_s to second order in $(u-U_s)$. To simplify expressions, we use the abbreviations defined in (40). One observes the property

$$W_{x,y} - W_{y,x} \geq 0 \quad \text{for} \quad x \geq y \text{ and } \xi \geq 0 \tag{D6}$$

with the consequence that

$$\sqrt{2xy + (x+y)\xi \pm 2W_{x,x}W_{y,y}} = W_{x,y} \pm W_{y,x}. \tag{D7}$$

The condition $\partial Z_{1,2}(u)/(\partial u) = 0$ leads to the equation

$$x + y - \frac{\xi}{u^2-1} + \sigma \frac{2u\sqrt{x\,y}}{\sqrt{u^2-1}} = 0, \quad \sigma = (+1) \text{ for } Z_1, \quad \sigma = (-1) \text{ for } Z_2. \tag{D8}$$

At first, the zero condition (D8) has 4 solutions, which are the same for both signs of σ:

$$U_1 = -\frac{W_{x,x} - W_{y,y}}{x-y}, \quad U_2 = -U_1, \quad U_3 = -\frac{W_{x,x} + W_{y,y}}{x-y}, \quad U_4 = -U_3. \tag{D9}$$

Only two of them are genuine solutions: (U_2, U_3) and (U_1, U_4) in the case $\sigma = 1$ and $\sigma = (-1)$, respectively. We remind that $x > y$, which implies that the two stationary points $U_2 > 1$ and $U_4 > 1$ lie on the integration path $u > 1$. So we approximate Z_1 near $u = U_2$ and Z_2 near $u = U_4$ as follows

$$Z_1(u) = Z_1(0) + \zeta_1(u-U_2)^2, \quad Z_2(u) = Z_2(0) + \zeta_2(u-U_4)^2, \tag{D10}$$

$$\begin{aligned} Z_1(0) &= -\mathbf{i}\,\frac{\pi}{4} + \frac{\mathbf{i}\,\lambda}{8\tau}\left[(x+y)\left(W_{x,x} - W_{y,y}\right) + 2\sqrt{x\,y}\left(W_{x,y} - W_{y,x}\right)\right] + \\ &\quad \Xi \ln\left(\Xi_1\right). \end{aligned} \tag{D11}$$

$$\begin{aligned} \zeta_1 &= \mathbf{i}\,\frac{\lambda(x-y)^4}{8\tau}\left\{\sqrt{xy}\,(x-y)^2 + \left(W_{x,x} - W_{y,y}\right)\left[\sqrt{xy}\left(W_{y,y} - W_{x,x}\right) + \right.\right. \\ &\quad \left.\left.\xi(W_{x,y} - W_{y,x})\right]\right\} \times \left[W_{x,y} - W_{y,x}\right]^{-5}. \end{aligned} \tag{D12}$$

$$\begin{aligned} Z_2(0) &= \mathbf{i}\,\frac{\pi}{4} + \frac{\mathbf{i}\,\lambda}{8\tau}\left[(x+y)(W_{x,x} + W_{y,y}) - 2\sqrt{x\,y}\left(W_{x,y} + W_{y,x}\right)\right] + \\ &\quad \Xi \ln\left(\Xi_2\right). \end{aligned} \tag{D13}$$

$$\begin{aligned} \zeta_2 &= \mathbf{i}\,\frac{\lambda(x-y)^4}{8\tau}\left\{\sqrt{xy}\left(W_{x,y} + W_{y,x}\right) + \xi(W_{x,x} + W_{y,y})\right\} \times \\ &\quad \left[W_{x,y} + W_{y,x}\right]^{-4}. \end{aligned} \tag{D14}$$

The saddle-point integration of $l_{asy}(u)$, is conventionally carried out (without error estimate). The integration path, in each of the two terms of (D4), is deformed such that it goes in the direction of steepest descend. In the first term of (D4), we apply the transformation $u \to v$ with $u = U_2 + v \exp(\mathbf{i}\,\phi)$, $v \in \mathbf{R}$, with the result

$$Z_1(u) \to Z_1(0) + \exp\left(\mathbf{i}\,(\theta + 2\phi)|\zeta_1|\, v^2\right), \quad \theta = \arg(\zeta_1) = \pi/2. \tag{D15}$$

The condition of steepest descent amounts to $\theta + 2\phi = \pi$, i.e. $\phi = \pi/4$. We get the Gaussian integral

$$\exp(\mathbf{i}\,\phi) \int_{-\infty}^{\infty} \mathrm{d}v \, \exp\left(-|\zeta_1|v^2\right) = \exp(\mathbf{i}\,\pi/4)\sqrt{\pi/|\zeta_1|}. \tag{D16}$$

The amplitude factor in (D4), $1/\sqrt{2\pi z}$, is taken at the corresponding stationary point $z_1 = z(U_2)$ and $z_2 = z(U_4)$, where

$$z_1 = \mathbf{i}\,\lambda A_s \sqrt{xy}\,\sqrt{U_2^2 - 1} = \frac{\lambda}{4\tau}\sqrt{xy}\,(W_{x,y} - W_{y,x})\,, \tag{D17}$$

$$z_2 = \mathbf{i}\,\lambda A_s \sqrt{xy}\,\sqrt{U_4^2 - 1} = \frac{\lambda}{4\tau}\sqrt{xy}\,(W_{x,y} + W_{y,x})\,. \tag{D18}$$

The integration of the Z_2 term works analogously. Combining results, the saddle-point integral, which corresponds to the integrand $l(u) \to l_{asy}(u)$ in (D2), is given by

$$L_{A_s}^{sp} = \frac{\exp(\mathbf{i}\,\pi/4)}{\sqrt{2 z_1 |\zeta_1|}} \exp\left(Z_1(0)\right) + \frac{\exp(\mathbf{i}\,\pi/4)}{\sqrt{2 z_2 |\zeta_2|}} \exp\left(Z_2(0)\right). \tag{D19}$$

It takes some efforts to bring $L_{A_s}^{sp}$ of (D19) into the simplified form $L_{A_s}^{sp}$ as given in (41). As a test, we form the difference, use the definitions in (40) for Ξ and $W_{u,v}$, and choose the integer parameters $x = 2, y = 1, \xi = 1$. By the command FullSimplify, Mathematica renders the exact zero result identically in $\lambda > 0$ and $\tau > 0$:

$$\text{FullSimplify}[(D19) - (41), \text{Assumptions} \to \lambda > 0 \;\text{ AND }\; \tau > 0] = 0. \tag{D20}$$

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