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On Justification of Superstatistics in Terms of Homotopy Theory

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Abstract

This paper investigates from the viewpoint of homotopy categories in Welded Tangleoids whether superstatistics is applicable. Boltzmann-Gibbs distributions (and Bose-Einstein for bosons or Fermi-Dirac for fermions in the quantum case) appear as the solution of kinetic equations (Boltzmann-Gibbs are stationary solution of classical Boltzmann equation and Fermi-Dirac or Bose-Einstein are stationary solution of the Quantum Boltzmann equation).

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1 Introduction

Recently, Effective Quantum Field Theories (EQFTs) were topologically characterized in terms of Welded Tangleoids. With effective quantum field theories, composite particles like atoms (consisting of electrons, protons and neutrons) and even macroscopic bodies can be described in the framework on Quantum field theory (QFT). Since it remains an open question whether particles regarded as elementary particles in our current understanding of particle physics might be composite, one may treat every QFT as an effective one. Theories that try to describe particles and interactions as elementary as possible like String theory have many open questions on its validity.

Recently, Effective Quantum Field Theories (EQFTs) were topologically characterized in terms of Welded Tangleoids. With effective quantum field theories, composite particles like atoms (consisting of electrons, protons and neutrons) and even macroscopic bodies can be described in the framework on Quantum field theory (QFT). Since it remains an open question whether particles regarded as elementary particles in our current understanding of particle physics might be composite, one may treat every QFT as an effective one. Theories that try to describe particles and interactions as elementary as possible like String theory have many open questions on its validity [dHD13].

Moreover, in recent times, generalizations of Statistical mechanics are requested. Traditionally, the fundamental pillar of Statistical mechanics is Boltzmann's formula for entropy

$$S = -k_B \log(W) \qquad (1)$$

with number of microstates W and Boltzmann's constant k_B. In the last decades a generalization of the Statistical definition of entropy (1) were proposed, e.g. Tsallis entropy [Tsa09, KLS05]. These apply if the system has long-ranged interactions, non-ergodicity or fractal structure of the phase space. Several justifications of generalized statistical mechanics were proposed for system that appear to have non-extensive thermodynamic behavior when treated with Boltzmann's formalism. One of them is called "Superstatistics" [HTGM11, Coh04]. This theory assumes that the local equilibrium state obeys the Boltzmann-Gibbs distribution

$$f(E-n) = \frac{1}{Z(\beta)} e^{-\beta E_n}, \quad Z(\beta) = \sum_n e^{-\beta E_n} \qquad (2)$$

where E_n is the n-th energy state and $\beta = \frac{1}{k_B T}$ with temperature T. But due to fluctuations of temperature over space and time, one may regard the effective energy distribution as another than the classical Boltzmann-Gibbs one. To account for spatio-temporal fluctuations in temperature, one introduces a probability distribution $P(\beta)$ that denotes the probability that the inverse temperature β will be observed. One gets the effective distribution f_{eff} by taking the average with this probability distribution over (2), thus:

$$f_{eff}(E_n) = \int_0^\infty \frac{P(\beta)}{Z(\beta)} e^{-\beta E_n} d\beta \qquad (3)$$

If there is full certainty about the observed temperature that is denoted by β_0 one has the distribution $P(\beta) = \delta(\beta - \beta_0)$ and Boltzmann-Gibbs distribution is recovered. In case of a χ^2-distribution of inverse temperature, Tsallis statistics

$$f_{eff}(E_n) = \frac{1}{Z}(1 - (q-1)\frac{E_n}{\beta_0})^{\frac{1}{q-1}} \qquad (4)$$

with parameters q, β_0 is recovered. This paper investigates from the viewpoint of homotopy categories in Welded Tangleoids whether superstatistics is applicable. Boltzmann-Gibbs distributions (and Bose-Einstein for bosons or Fermi-Dirac for fermions in the quantum case) appear as the solution of kinetic equations [Ver20](Boltzmann-Gibbs are stationary solution of classical Boltzmann equation and Fermi-Dirac or Bose-Einstein are stationary solution of the Quantum Boltzmann equation). We will, for simplicity focus on Boltzmann statistics, since it is also the limiting case of Fermi-Dirac or Bose-Einstein distribution for sufficiently high temperatures and low particle or quasiparticle density. Also, only fluctuations in inverse temperature are addressed. Since EQFTs form the basis of macroscopic objects which are commonly described by classical physics, semi-classical treatments, here with the assumption that the classical Boltzmann-Gibbs distribution is the stationary distribution for a kinetic equation, are plausible. The derivation of superstatistics formalism (3) from first principles will be considered in this paper.

The paper organization is as follows; In Section 2 we review the unoriented welded tangleoid categories. Then in Section 3 we givesup[erstatistic in terms of homotopy theory.

2 Unoriented Welded Tangle-oids

In this section we review the definition of welded tangle-oids categories defined in [Alb22]. A monoidal category (see for example [Kas12]) of unoriented welded tangle-oids have defined by giving a presentation by using the presentation of slideable $\frac{1}{2}$-monoidal categories [Alb22].

Definition 2.1. *[Alb22, definition 7.2.1] Consider the monoidal graph*

$$\beta = (\mathbb{N}, E(\beta), \otimes_0, 0, \delta_1, \delta_2),$$

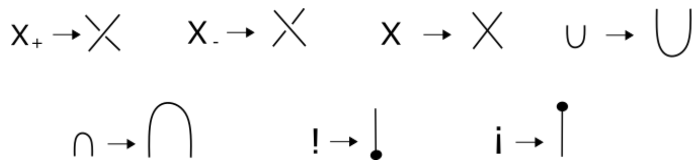
where for all $m, n \in \mathbb{N}$, $m \otimes_0 n = m + n$, *and*

$$E(\beta) = \{X_+, X_-, X, \cup, \cap, ¡, !\},$$

the incidence maps

$$\begin{array}{rrrr} \delta_1 X_+ = 2, & \delta_2 X_+ = 2, & \delta_1 X_- = 2, & \delta_2 X_- = 2, \\ \delta_1 X = 2, & \delta_2 X = 2, & \delta_1 \cup = 0, & \delta_2 \cup = 2, \\ \delta_1 \cap = 2, & \delta_2 \cap = 0, & \delta_1 ¡ = 1, & \delta_2 ¡ = 0, \\ \delta_1 ! = 0, & \delta_2 ! = 1. & & \end{array}$$

These generators can be presented geometrically as



Consider the path category, see for example ([Hig71], over β^*, the extent of the monoidal graph β.

$$P(\beta^*) = (\mathbb{N}, \hom_{P(\beta^*)}(n, m), \bullet, \phi_{\_}).$$

Therefore

$$\Omega(\beta) = (P(\beta^*), \otimes_0, 0, {}_n\#, \#_m)$$

is a $\frac{1}{2}$-monoidal category, whose set of objects is the set of natural numbers, where for all $n, m, k \in \mathbb{N}$;

$${}_n\#_m(k) = n \otimes_0 k \otimes_0 m = n + k + m,$$

and for all generating morphism $(f : k \to k') \in E(\beta)$, we have

$${}_n\#_m(f) = n + k + m \xrightarrow{n \Theta f \Theta m} n + k' + m.$$

Then we have the free-$\frac{1}{2}$-monoidal category-triple

$$(\beta, \Omega(\beta), \delta).$$

Definition 2.2 (Unoriented welded tangle-oids category)**.** *The* unoriented welded tangle-oids category $UWTC$ *is the strict monoidal category formally presented by*

$$\mathfrak{F}\Big(\Omega(\beta) \Big/ \underline{\overline{W}}\Big),$$

where $\Omega(\beta)$ *defined in [Alb22, Section 7.2] and* $\overline{W}$ *is the* $\frac{1}{2}$*-monoidal closure of the congruence template* W *that is defined as follows.*

Given $m, n \in \mathbb{N}$*, then* $W_{m,n}$ *is the relation in* $\hom_{P(\beta^*)}(m,n)$*, defined as (the picture will follow)*

In $\hom_{P(\beta^*)}(1,1)$*, we have the only relations*

- $[WT_1]$ *:* $(\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1)$.
- $[WT_2]$: $(\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_3]$: $(\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1)$.
- $[WT_4]$: $(\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup) \sim_{W_{1,1}} \mathrm{id}_1 \sim_{W_{1,1}} (\mathrm{id}_1 \otimes \cap)(\cup \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(2,2)$*, we have the only relation*

- $[WT_5]$: $X_- X_+ \sim_{W_{2,2}} \mathrm{id}_2 \sim_{W_{2,2}} X_+ X_-$.

In $\hom_{P(\beta^*)}(3,3)$*, we have the only relations*

- $[WT_6]$: $(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_7]$: $(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X)(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)$.
- $[WT_8]$: $(X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1) \sim_{W_{3,3}} (\mathrm{id}_1 \otimes X_+)(X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X)$.

In $\hom_{P(\beta^*)}(3,1)$*, we have the only relations*

- $[WT_9]$: $(\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_-) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_+ \otimes \mathrm{id}_1)$.
- $[WT_9]'$: $(\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X_+) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X_- \otimes \mathrm{id}_1)$.
- $[WT_9]''$: $(\cap \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes X) \sim_{W_{3,1}} (\mathrm{id}_1 \otimes \cap)(X \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(1,3)$*, we have the only relations*

- $[WT_{10}]$: $(\mathrm{id}_1 \otimes X_+)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_- \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]'$: $(\mathrm{id}_1 \otimes X_-)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X_+ \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.
- $[WT_{10}]''$: $(\mathrm{id}_1 \otimes X)(\cup \otimes \mathrm{id}_1) \sim_{W_{1,3}} (X \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes \cup)$.

In $\hom_{P(\beta^*)}(1,0)$*, we have the only relation*

- $[WT_{11}]$: $\cap(\mathrm{id}_1 \otimes !) \sim_{W_{1,0}} ¡ \sim_{W_{1,0}} \cap(! \otimes \mathrm{id}_1)$.

In $\hom_{P(\beta^*)}(0,1)$*, we have the only relation:*

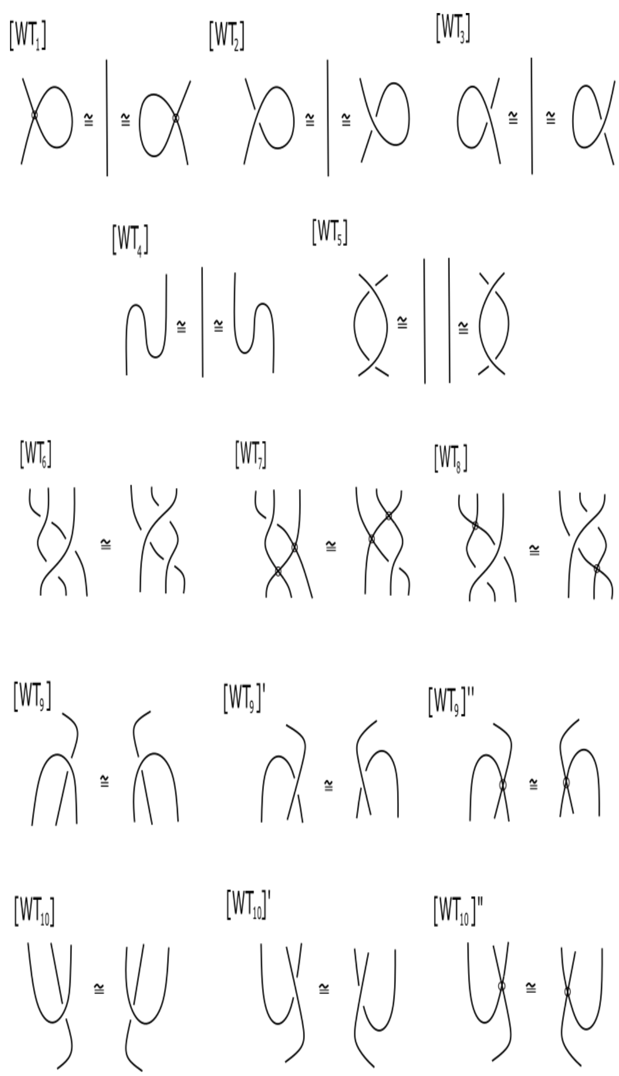
- $[WT_{12}] : (\mathrm{id}_1 \otimes ¡)\cup \sim_{W_{0,1}} ! \sim_{W_{0,1}} (¡ \otimes \mathrm{id}_1)\cup.$

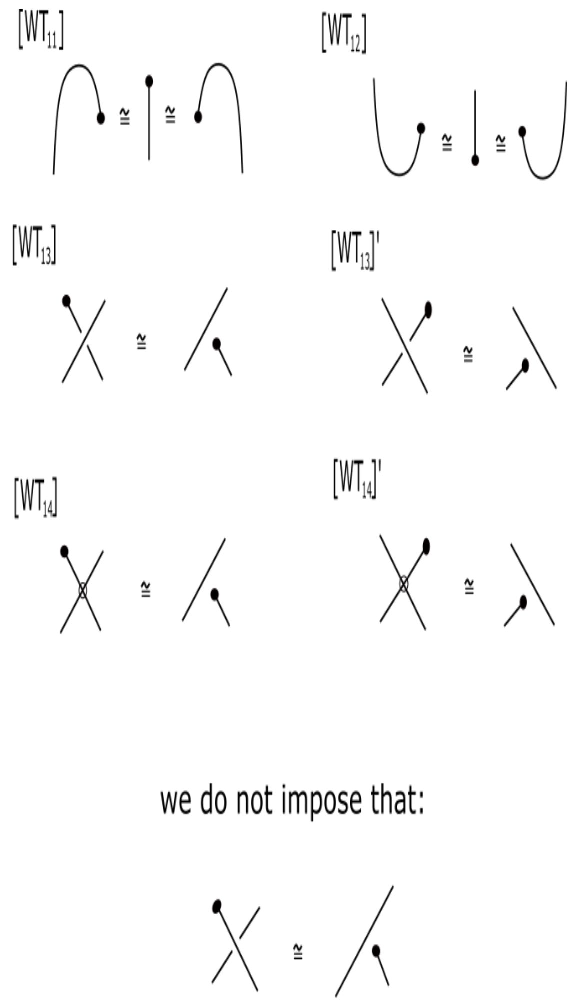
In $\hom_{P(\beta^*)}(2,1)$*, we have the only relations*

- $[WT_{13}] : (¡ \otimes \mathrm{id}_1)X_+ \sim_{W_{2,1}} \mathrm{id}_1 \otimes ¡.$
- $[WT_{13}]' : (\mathrm{id}_1 \otimes ¡)X_- \sim_{W_{2,1}} ¡ \otimes \mathrm{id}_1.$
- $[WT_{14}] : (¡ \otimes \mathrm{id}_1)X \sim_{W_{2,1}} \mathrm{id}_1 \otimes ¡.$
- $[WT_{14}]' : (\mathrm{id}_1 \otimes ¡)X \sim_{W_{2,1}} ¡ \otimes \mathrm{id}_1.$

Note that we do not impose that in $\hom_{P(\beta^*)}(2,1)$***:***
$(¡ \otimes \mathrm{id}_1)X_- \nsim_{W_{2,1}} \mathrm{id}_1 \otimes ¡.$

These relations can be present geometrically as (note we read the diagram from bottom to top)





3 Justification of Superstatistics in terms of homotopy theory

Consider a homotopy from the boundary $\partial\Gamma$ of the welded tangleoid over an interval to the interior of the tangleoid . With tangleoids, composite particle and quasiparticle scattering is depicted. The homotopy map has the form:

$$h : \partial\Gamma \times [0, 1] \to \Gamma. \qquad (5)$$

We choose the boundary $\partial\Gamma$ to be on one equal-time surface. We will pick this equal-time surface as a local kinetic equilibrium state where Boltzmann-Gibbs statistics holds. On other instants of time, the Boltzmann-Gibbs distribution is deformed due to nonequilibrium effects e.g. viscous stress or heat conduction.

We can pick a collection of multiple points (x, s) with $x \in \partial\Gamma$ and $s \in [0, 1]$ that satisfy Boltzmann-Gibbs statistics. Within the neighborhood of one point, denoted by $N(x, s)$ we have nonequilibrium behavior. Gluing all neighborhoods together, we cover the space $\partial\Gamma \times [0, 1]$ which is the domain of the homotopy map h. Define a functor F from homotopy categories (with homotopy spaces as objects, gluing and homotopies as morphisms) to function space category (with functions as objects and linear operators as morphisms). This functor transforms the points (x, s) to its local Boltzmann distributions and the gluing operation to integration. Homotopies are regarded as homeomorphisms in $\partial\Gamma \times [0, 1]$ that are transformed into the adjustment of the support within the function space. In total the glued space $\cup_{x\in\partial\Gamma,s\in[0,1]}(x, s)$ will get the following transformation under the action of F:

$$F(\cup_{x\in\partial\Gamma,s\in[0,1]}(x, s)) = \sum_{(x,t)\in\Omega} \frac{e^{-\beta(x,t)E_n}}{Z(\beta(x, t))}, \qquad (6)$$

where Ω is the spacetime manifold with its points (x, t). Using the functor on the homotopy operation and extending the support on the right hand side of (6), we can write:

$$\begin{aligned} &F(h(\cup_{x\in\partial\Gamma,s\in[0,1]}(x, s))) = F(\Gamma) \\ &= \int_0^\infty d\beta \int dt d^3x \delta(\beta - \beta(x, t)) \frac{e^{-\beta(x,t)E_n}}{Z(\beta(x, t))} \end{aligned} \qquad (7)$$

Finally, defining $\int dt d^3x \delta(\beta - \beta(x, t)) = P(\beta)A$ with some normalization constant A leads to the effective energy distribution function (3) in a way that

$$F(\Gamma) = f_{eff}(E_n). \qquad (8)$$

Therefore, we have identified the effective equilibrium distribution with the physical behavior that appears during the depicted welded tangleoid and its homotopy.

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