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Wormholes Supported by Small Extra Dimensions

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Abstract

Holding a Morris-Thorne wormhole open requires a violation of the null energy condition, calling for the need for so-called exotic matter near the throat. Many researchers consider exotic matter to be completely unphysical in classical general relativity. It has been shown, however, that the existence of an extra macroscopic dimension can resolve this issue: the throat could be lined with ordinary matter, while the extra dimension is then responsible for the unavoidable energy violation. The purpose of this paper is to show that the extra dimension can be microscopic, a result that is consistent with string theory.

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1 Introduction

This paper is concerned with a number of fundamental issues in the study of Morris-Thorne wormholes, even raising the question whether a basic wormhole structure can even be hypothesized. Moreover, while they may be just as good a prediction of Einstein's theory as black holes, wormholes are subject to severe restrictions from quantum field theory, in particular, the need to violate the null energy condition, calling for the existence of "exotic matter" to hold a wormhole open. It has been shown, however, that this requirement can be met via the existence of an extra macroscopic spatial dimension. It is proposed in this paper that the extra dimension can be extremely small, an appoach that is consistent with string theory. Furthermore, the existence of the extra dimension would allow the throat of the wormhole to be lined with ordinary matter, while the unavoidable violation of the null energy condition can be attributed to the higher spatial dimension.

The other issue to be addressed is the enormous radial tension at the throat of any moderately-sized wormhole, a problem that is usually ignored.

2 Wormhole structure

Wormholes have been a subject of interest ever since it was realized that the Schwarzschild solution and therefore black holes can be viewed as wormholes, albeit nontraversable. More recently, the problem of entanglement has drawn attention to a special type of wormhole, the Einstein-Rosen bridge, to explain this phenomenon. Accordingly, we will assume that a basic wormhole structure can be hypothesized.

While there had been some forerunners, what came to be called a Morris-Thorne wormhole was first proposed by Morris and Thorne [1], who proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where

$$e^{2\lambda(r)} = \frac{1}{1 - \frac{b(r)}{r}}. \tag{2}$$

(We are using units in which $c = G = 1$.) The terminology introduced in Ref. [1] has become standard: $\Phi = \Phi(r)$ is called the *redshift function*; this function must be finite everywhere to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole whenever it is depicted in an embedding diagram [1]. The spherical surface $r = r_0$ is called the *throat* of the wormhole. In a Morris-Thorne wormhole, the shape function must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$, and $b'(r_0) < 1$, called the *flare-out condition* in Ref. [1]. In classical general relativity, the flare-out condition can only be met by violating the null energy condition (NEC), which states that for the energy-momentum tensor $T_{\alpha\beta}$

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0 \text{ for all null vectors } k^\alpha. \tag{3}$$

Matter that violates the NEC is called "exotic" in Ref. [1]; the term is borrowed from quantum mechanics. To see the effect on wormholes, consider the radial outgoing null vector $(1, 1, 0, 0)$, which yields

$$T_{\alpha\beta}k^{\alpha}k^{\beta} = \rho + p_r < 0 \tag{4}$$

whenever the NEC is violated. Here $T^t{}_t = -\rho(r)$ is the energy density, $T^r{}_r = p_r(r)$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t(r)$ is the lateral (transverse) pressure. Another requirement is *asymptotic flatness:*

$$\lim_{r\to\infty}\Phi(r) = 0 \quad \text{and} \quad \lim_{r\to\infty}\frac{b(r)}{r} = 0. \tag{5}$$

For later reference, let us now state the Einstein field equations in the orthonormal frame:

$$G_{\hat{\alpha}\hat{\beta}} = R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}Rg_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}, \tag{6}$$

where

$$g_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

In the orthonormal frame, we can simply write $T_{00} = \rho$ and $T_{11} = p_r$.

3 The exotic-matter problem

As noted in the Introduction, holding a wormhole open requires a violation of the NEC, thereby calling for the existence of exotic matter, at least in the vicinity of the throat. The problematical nature of exotic matter in classical general relativity has led to a certain skepticism: many researchers consider such wormhole solutions to be completely unphysical, thereby ruling out the existence of macroscopic traversable wormholes in Einstein's theory. This has suggested solutions beyond the classical theory. For example, it was proposed by Lobo and Oliveira [2] that in $f(R)$ modified gravity, the wormhole throat could be lined with ordinary matter, while the violation of the NEC can be attributed to the higher-order curvature terms. Another possibility is to invoke noncommutative geometry, an offshoot of string theory [3, 4].

In this paper, we are going to be more interested in the effects of an extra spatial dimension, an example of which is the *induced-matter theory* by P.S. Wesson [5, 6, 7]: what we perceive as matter is merely the impingement of the higher-dimensional space onto ours. The relationship between the energy violation and the existence of extra dimensions is taken up in Ref. [8], where the extra dimensions are assumed to be macroscopic. String theory, on the other hand, assumes the existence of extra small dimensions; these are sometimes referred to as "compactified" or "curled up." So it makes sense for us to follow suit and assume the existence of at least one small extra dimension.

4 A small extra dimension

If we are going to assume the existence of an extra dimension, we must first decide on the line element. With Eq. (1) in mind, let us consider the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{2\mu(r,l)}dl^2, \tag{8}$$

where l is the extra coordinate. This choice was motivated in part by symmetry considerations: all the exponential terms have the same form.

Our next step is to list the components of the Ricci tensor from Ref. [8] in the orthonormal frame:

$$\begin{aligned} R_{00} = &-\frac{1}{2}\frac{d\Phi(r)}{dr}\frac{rb'-b}{r^2} + \frac{d^2\Phi(r)}{dr^2}\left(1-\frac{b}{r}\right) \\ &+\left[\frac{d\Phi(r)}{dr}\right]^2\left(1-\frac{b}{r}\right) + \frac{2}{r}\frac{d\Phi(r)}{dr}\left(1-\frac{b}{r}\right) + \frac{d\Phi(r)}{dr}\frac{\partial\mu(r,l)}{\partial r}\left(1-\frac{b}{r}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} R_{11} = &\frac{1}{2}\frac{d\Phi(r)}{dr}\frac{rb'-b}{r^2} - \frac{d^2\Phi(r)}{dr^2}\left(1-\frac{b}{r}\right) \\ &-\left[\frac{d\Phi(r)}{dr}\right]^2\left(1-\frac{b}{r}\right) + \frac{rb'-b}{r^3} - \frac{\partial^2\mu(r,l)}{\partial r^2}\left(1-\frac{b}{r}\right) \\ &+\frac{1}{2}\frac{\partial\mu(r,l)}{\partial r}\frac{rb'-b}{r^2} - \left[\frac{\partial\mu(r,l)}{\partial r}\right]^2\left(1-\frac{b}{r}\right), \end{aligned} \tag{10}$$

$$R_{22} = R_{33} = -\frac{1}{r}\frac{d\Phi(r)}{dr}\left(1-\frac{b}{r}\right) + \frac{1}{2}\frac{rb'-b}{r^3} + \frac{b}{r^3} - \frac{1}{r}\frac{\partial\mu(r,l)}{\partial r}\left(1-\frac{b}{r}\right), \tag{11}$$

and

$$\begin{aligned} R_{44} = &-\frac{d\Phi(r)}{dr}\frac{\partial\mu(r,l)}{\partial r}\left(1-\frac{b}{r}\right) - \frac{\partial^2\mu(r,l)}{\partial r^2}\left(1-\frac{b}{r}\right) \\ &+\frac{1}{2}\frac{\partial\mu(r,l)}{\partial r}\frac{rb'-b}{r^2} - \left[\frac{\partial\mu(r,l)}{\partial r}\right]^2\left(1-\frac{b}{r}\right) - \frac{2}{r}\frac{\partial\mu(r,l)}{\partial r}\left(1-\frac{b}{r}\right). \end{aligned} \tag{12}$$

The Ricci tensor plays a key role in analyzing the wormhole solution. Upon closer examination, it turns out that the function $\mu(r,l)$ never occurs as an exponent or as a factor, but only as a derivative. So the magnitude of $\mu(r,l)$ does not have an effect: what matters is the rate of change of $\mu(r,l)$ with respect to r. As far as the Ricci tensor is concerned, $\mu(r,l)$ can have any magnitude and either algebraic sign.

Returning to line element (8), it now follows that if $\mu(r,l)$ is negative and large in absolute value, then $e^{\mu(r,l)}$ can be extremely small or even compactified (in the language of string theory). So our basic assumption, the existence of an extra small spatial dimension, is consistent with string theory. The reason is that the small size is the only property from string theory that we are making use of. This is going to lead to our main conclusion.

5 The large radial tension

Before continuing, let us to return to Ref. [1] to consider another problem, the radial tension at the throat. First we need to recall that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$. According to Ref. [1], the Einstein field equations can be rearranged to yield $\tau(r)$. Temporarily reintroducing c and G, we obtain

$$\tau(r) = \frac{b(r)/r - 2[r - b(r)]\Phi'(r)}{8\pi G c^{-4} r^2}. \tag{13}$$

The radial tension at the throat therefore becomes

$$\tau(r_0) = \frac{1}{8\pi G c^{-4} r_0^2} \approx 5 \times 10^{41} \frac{\mathrm{dyn}}{\mathrm{cm}^2} \left(\frac{10\,\mathrm{m}}{r_0}\right)^2. \tag{14}$$

As noted in Ref. [1], for a throat size of $r_0 = 3\,\mathrm{km}$, $\tau(r)$ has the same magnitude as the pressure at the center of a massive neutron star. This is enough to suggest that moderately-sized wormholes are actually compact stellar objects [9]. According to Eq. (14), however, wormholes with low tidal forces could only exist on very large scales, i.e., such wormholes require very large throat sizes.

6 The main result

We know from Sec. 2 that the NEC states that for the energy-momentum tensor $T_{\alpha\beta}$, $T_{\alpha\beta}k^\alpha k^\beta \geq 0$ for all null vectors k^α. We also recall that an ordinary Morris-Thorne wormhole [line element (1)] can only be maintained if the NEC is violated. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation reads

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \tag{15}$$

Returning to Eq. (6), observe that

$$8\pi(\rho + p_r) = 8\pi(T_{00} + T_{11}) = \left[R_{00} - \frac{1}{2}R(-1)\right] + \left[R_{11} - \frac{1}{2}R(1)\right] = R_{00} + R_{11}. \tag{16}$$

Since we are primarily interested in the vicinity of the throat, we assume that $1 - b(r_0)/r_0 = 0$. So it follows immediately from Eqs. (9) and (10) that

$$8\pi(\rho+p_r)|_{r=r_0} = \frac{rb'-b}{r^3} + \frac{1}{2}\frac{\partial(\mu(r,l)}{\partial r}\frac{rb'-b}{r^2}\bigg|_{r=r_0} = \frac{b'(r_0)-1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial\mu(r_0,l)}{\partial r}\right]. \tag{17}$$

Recalling that $b'(r_0) < 1$, we obtain

$$\rho + p_r > 0 \quad \text{at} \quad r = r_0 \tag{18}$$

provided that

$$\frac{\partial\mu(r_0,l)}{\partial r} < -\frac{2}{r_0}. \tag{19}$$

So, thanks to the extra dimension, the NEC is satisfied at the throat, which can therefore be lined with ordinary matter. It is interesting to note that if $\mu(r,l)$ is independent of r, so that $\partial\mu(r,l)/\partial r = 0$, we get

$$\rho + p_r|_{r=r_0} = \frac{1}{8\pi}\frac{b'(r_0)-1}{r_0^2} < 0, \tag{20}$$

the usual condition for a Morris-Thorne wormhole.

Condition (19) also plays a key role in maintaining the wormhole. To show this, consider the null vector $(1, 0, 0, 0, 1)$. Assuming that the Einstein field equations hold in the five-dimensional spacetime, we now have

$$G_{00}+G_{44} = 8\pi(T_{00}+T_{44}) = \left[R_{00} - \frac{1}{2}Rg_{00}\right] + \left[R_{44} - \frac{1}{2}Rg_{44}\right] = R_{00}+R_{44} \tag{21}$$

and given that $1 - b(r_0)/r_0 = 0$, we get from Eqs. (9) and (12) that

$$R_{00} + R_{44} = \frac{1}{2}\frac{rb'-b}{r^2}\left[-\frac{d\Phi(r)}{dr} + \frac{\partial\mu(r,l)}{\partial r}\right]. \tag{22}$$

Since $\partial\mu(r_0,l)/\partial r < -2/r_0$, the second factor on the right side of Eq. (22) is positive if

$$\frac{d\Phi(r_0)}{dr} = -A < \frac{\partial\mu(r_0,l)}{\partial r} < -\frac{2}{r_0}, \tag{23}$$

which is similar to Condition (19). It follows that $\rho+p_r|_{r=r_0} < 0$. We conclude that the NEC is satisfied at the throat in the four-dimensional spacetime but violated in the five-dimensional spacetime.

To finish the discussion, we need to return to Sec. 5 and recall that the wormhole we are considering can only exist on a very large scale, i.e., with a very large throat radius $r = r_0$. Inequality (19) therefore implies that $\partial\mu(r_0,l)/\partial r$ is close to zero, and hence

$$\frac{\partial}{\partial r}e^{\mu(r_0,l)} = e^{\mu(r,l)}\,\frac{\partial\mu(r,l)}{\partial r}\bigg|_{r=r_0} < e^{\mu(r_0,l)}\left(-\frac{2}{r_0}\right) \tag{24}$$

is also close to zero. So from the perspective of the four-dimensional spacetime, the extra dimension is not only nonincreasing, the small absolute value of the derivative in (24) actually implies that the extra dimension is essentially constant. In summary, a sufficiently large throat size ensures that the radial tension remains low and that the size of the extra dimension remains fixed.

Remark: For the NEC to be met near the throat in the four-dimensional case, the shape function has to meet the condition $b'(r_0) > 1/3$. [See Ref. [8] for details.]

7 Conclusion

It has been argued that wormholes are just as good a prediction of Einstein's theory as black holes, but they are subject to severe restrictions from quantum field theory. In particular, to hold a wormhole open requires a violation of the NEC, which means that the throat of the wormhole has to be lined with "exotic matter." The problematical nature of exotic matter has led many researchers to conclude that such wormhole solutions are completely unphysical in classical general relativity, thereby ruling out the existence of macroscopic traversable wormholes. It has been shown, however, that the existence of an extra macroscopic spatial dimension can account for the unavoidable violation of the NEC, while allowing the throat of the wormhole to be constructed from ordinary matter. The purpose of this paper is to show that the extra dimension can be microscopic. This result is consistent with string theory which assumes that the extra dimensions are "compactified" or "curled up." Since the small size is the only property from string theory that we are making use of, our result suggests that string theory is able to support traversable wormholes.

Regarding the redshift function $\Phi = \Phi(r)$, in the original Morris-Thorne wormhole, this function can be freely assigned. In our situation, however, the need to violate the NEC has led to Condition (22), implying that $\Phi(r)$ and $\mu(r, l)$ have to meet similar conditions, namely Inequalities (19) and (23), respectively. Here the throat radius $r = r_0$ is very large, thereby implying that traversable wormholes can only exist on very large scales.

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