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Equations of Electrodynamics in the Context of the Brans-Dicke Theory

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Abstract

We consider the Brans-Dicke theory of gravity, assuming a weak gravitational field generated by a material source with a rotating motion. In this context, we find the equations of electrodynamics, verifying the gravitational effects, such as gravitomagnetic effects, that contribute to the production of electric and magnetic fields. The equations obtained are compared with those predicted by general relativity. Then, some particular results are discussed.

Keywords: Electrodynamics, Weak Field Approximation, Brans-Dicke Theory.

1 Introduction

We can obtain the equations of electrodynamics in a curved spacetime [10], and the most varied applications can be made [1, 2, 7]. It is interesting to observe the presence of new terms in these equations, compared to Maxwell's equations of flat spacetime. These may be additional terms that depend on mass, for example. Also, it is worth noting the case where the material source, which determines the geometric properties of a given spacetime, is rotating; if the weak field approximation is admitted, one shows that the rotation of a mass creates the gravitomagnetic field [8]. Then, there should appear terms in the equations for the electric and magnetic fields that depend on the angular momentum of the rotating source.

On the other hand, the theory of general relativity is the standard theory of gravity. However, there are also some alternative theories, such as the scalar-tensor theories [16]. These theories generalize Einstein's theory, since gravitational effects are described by the spacetime metric and also by a scalar field ϕ, which can be related to the variation of Newtonian gravitational constant. In the scalar-tensor theories context, a coupling parameter $\omega = \omega(\phi)$ of the scalar field with the geometry is included. When $\omega = constant$, one have the case of the Brans-Dicke theory [6], being that the value of ω must be fixed from experimental observations. The scalar-tensor theories utilize some ingredients of string theories, such as a dilaton-like gravitational scalar field and its non-minimal coupling to the curvature [12]. Furthermore, many other aspects are being studied in these theories [4, 13, 3].

In this paper, we will consider the Brans-Dicke scalar-tensor theory, supposing a weak gravitational field generated by a material source with a rotating motion. Using the fact that one can establish a straightforward correspondence between weak field solutions in general relativity and Brans-Dicke theory [5], we will determine the equations of electrodynamics in Brans-Dicke theory. After, some specific cases will be explored.

The paper is organized as follows. In Section 2, we consider the weak field approximation of general relativity and the equations of electrodynamics are exhibited. Next, in Section 3, we obtain the appropriate metric in the Brans-Dicke theory from the metric of general relativity, subsequently carrying out the calculation of the equations of electrodynamics in this context. Some results are presented in Section 4. Finally, in Section 5, our conclusions are exposed.

2 The equations of electrodynamics in general relativity

Let us consider the weak field approximation of general relativity, in which the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ a small perturbation, such that only first-order terms in $h_{\mu\nu}$ are kept. In this approximation, the Kerr metric can be written as [11]

$$\begin{aligned} ds^2 &= c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 - \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j \\ &\quad + \frac{4}{c}(\overrightarrow{A} \cdot d\overrightarrow{r})dt, \end{aligned} \tag{2}$$

where c is the speed of light, $\Phi = GM/r$, G is the Newtonian gravitational constant and M is the central body mass. We use a Cartesian-like coordinate system $x^\alpha = (ct, \overrightarrow{r})$ with $\overrightarrow{r} = (x, y, z)$ and $\alpha = 0, 1, 2, 3$. Also, we define

$$\overrightarrow{A} = \frac{G(\overrightarrow{j} \times \overrightarrow{r})}{cr^3}, \tag{3}$$

being $\overrightarrow{j} = j\widehat{z}$ the angular momentum of the central body. In these conditions, we obtain

$$\sqrt{-g} = 1 + 2\frac{\Phi}{c^2}, \tag{4}$$

where g is the determinant of the metric.

Now, If we admit (2) as background metric, one can find the equations for the electric and magnetic fields, which sources are ρ and $\overrightarrow{J}$, the charge and current densities, respectively. For this, let us consider the field equations [10]

$$F_{\mu\nu;\sigma} + F_{\sigma\mu;\nu} + F_{\nu\sigma;\mu} = 0, \tag{5}$$

$$F^{\mu\nu}{}_{;\nu} = -\frac{4\pi}{c} J^\mu = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}F^{\mu\nu}\right)_{,\nu}, \tag{6}$$

and the current four-vector

$$J^\mu = \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^\mu}{dx^0} = \left(1 + \frac{\Phi}{c^2}\right) (\rho c, \overrightarrow{J}), \tag{7}$$

being $F_{\mu\nu}$ the electromagnetic field tensor, which is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{8}$$

Thus, the following expressions can be obtained in a straightforward way from the equations (5)-(8):

$$\nabla \cdot \overrightarrow{E} = 4\pi\rho \left(1 + \frac{\Phi}{c^2}\right) + \frac{2}{c^2}\overrightarrow{g} \cdot \overrightarrow{E} + \frac{2}{c^2}\nabla \cdot \left(\overrightarrow{B} \times \overrightarrow{A}\right), \tag{9}$$

$$\nabla \cdot \overrightarrow{B} = 0, \tag{10}$$

$$\nabla \times \overrightarrow{E} = -\frac{1}{c}\frac{\partial \overrightarrow{B}}{\partial t}, \tag{11}$$

$$\begin{aligned} \nabla \times \overrightarrow{B} \quad &= \quad \frac{4\pi}{c}\overrightarrow{J} \left(1 + \frac{5\Phi}{c^2}\right) + \frac{1}{c}\frac{\partial \overrightarrow{E}}{\partial t} \left(1 + \frac{4\Phi}{c^2}\right) \\ &\quad -\frac{2}{c^2}\overrightarrow{g} \times \overrightarrow{B} + \frac{2}{c^2}\nabla \times \left(\overrightarrow{E} \times \overrightarrow{A}\right) - \frac{2}{c^3}\frac{\partial \overrightarrow{B}}{\partial t} \times \overrightarrow{A}. \end{aligned} \tag{12}$$

In the above formulas, we utilize the usual expressions for the divergence and the curl of a vector in a Cartesian coordinate system. Also, we define $\overrightarrow{g} = -\nabla\Phi$. The equations (9)-(12) are the equations of electrodynamics in a spacetime generated by a rotating body.

3 The equations of electrodynamics in Brans-Dicke theory

In the sequence, the same problem will be considered in the context of the Brans-Dicke theory of gravity [6]. When we adopt the weak-field approximation, the solutions of Brans-Dicke theory can be obtained directly from the corresponding solutions in general relativity in a simple way [5]. In fact, a solution for Brans-Dicke can be written as

$$ds^2_{BD} = [1 - \varepsilon G_0]\, ds^2_{RG}\,(G \to G_0)\,, \tag{13}$$

where $G_0 = \left(\frac{2\omega+3}{2\omega+4}\right) G$ and $ds^2_{RG}\,(G \to G_0)$ represents the solution of general relativity to the corresponding problem with the exchange of G by G_0, with the scalar field $\phi(x) = \phi_0 + \varepsilon(x)$, being ϕ_0 a constant.

Then, using (2) and (13), we obtain

$$\begin{aligned} ds^2_{BD} &= [1 - \varepsilon G_0]\,[c^2\left(1 - 2\frac{\Phi_{BD}}{c^2}\right)dt^2 - \left(1 + 2\frac{\Phi_{BD}}{c^2}\right)\delta_{ij}dx^i dx^j \\ &\quad + \frac{4}{c}(\overrightarrow{A}_{BD}\cdot d\overrightarrow{r})dt], \end{aligned} \tag{14}$$

with $\Phi_{BD} = G_0 M/r$ and $\overrightarrow{A}_{BD} = [G_0(\overrightarrow{j} \times \overrightarrow{r})]/cr^3$. Still, it should be done $\varepsilon = 2M/c^2 r(2\omega+3)$ [6]. Thus, we have

$$\begin{aligned} ds^2_{BD} &= c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 - \left[1 + 2\left(\frac{\omega+1}{\omega+2}\right)\frac{\Phi}{c^2}\right]\delta_{ij}dx^i dx^j \\ &\quad + \frac{4}{c}\left(\frac{2\omega+3}{2\omega+4}\right)(\overrightarrow{A}\cdot d\overrightarrow{r})dt. \end{aligned} \tag{15}$$

Now, we find that

$$\sqrt{-g} = 1 + \left(\frac{2\omega+1}{\omega+2}\right)\frac{\Phi}{c^2}. \tag{16}$$

Repeating the procedure of the previous section, using (5)-(8), we can get the equations for the electric and magnetic fields

$$\begin{aligned} \nabla\cdot\overrightarrow{E} &= 4\pi\rho\left[1 + \frac{\omega\Phi}{(\omega+2)c^2}\right] + \left(\frac{2\omega+3}{\omega+2}\right)\frac{\overrightarrow{g}\cdot\overrightarrow{E}}{c^2} \\ &\quad + \frac{2}{c^2}\left(\frac{2\omega+3}{2\omega+4}\right)\nabla\cdot\left(\overrightarrow{B}\times\overrightarrow{A}\right), \end{aligned} \tag{17}$$

$$\nabla \cdot \overrightarrow{B} = 0, \tag{18}$$

$$\nabla \times \overrightarrow{E} = -\frac{1}{c}\frac{\partial \overrightarrow{B}}{\partial t}, \tag{19}$$

$$\begin{aligned} \nabla \times \overrightarrow{B} &= \frac{4\pi}{c}\overrightarrow{J}\left[1 + \frac{(5\omega+6)\Phi}{(\omega+2)c^2}\right] - \left(\frac{2\omega+3}{\omega+2}\right)\frac{\overrightarrow{g} \times \overrightarrow{B}}{c^2} \\ &+ \frac{2}{c^2}\left(\frac{2\omega+3}{2\omega+4}\right)\nabla \times \left(\overrightarrow{E} \times \overrightarrow{A}\right) + \frac{1}{c}\left[1 + \frac{(4\omega+6)\Phi}{(\omega+2)c^2}\right]\frac{\partial \overrightarrow{E}}{\partial t} \\ &- \frac{2}{c^3}\left(\frac{2\omega+3}{2\omega+4}\right)\frac{\partial \overrightarrow{B}}{\partial t} \times \overrightarrow{A}. \end{aligned} \tag{20}$$

Examining the equations (17) and (20), we note the participation of fields Φ, $\overrightarrow{g}$ and $\overrightarrow{A}$ generating new terms, which are absent when we find the Maxwell equations in Minkowski's spacetime. The discrepancy between the gravitational theories of Einstein and Brans-Dicke is encoded in the terms in which the constant ω appears, representing the influence of the scalar field ϕ. It is interesting to note that, in the limit $\omega \to \infty$, the equations (17) and (20) are reduced to the equations (9) and (12) [14, 9].

4 Some interesting results

Now, let us consider some results from the equations obtained in section 3.

Initially, let us admit that the magnetic field is a first-order term. Also, let $\overrightarrow{E}$ and $\overrightarrow{B}$ be static fields and $\overrightarrow{J} = 0$. Thus, the following equations are obtained from (17)-(20):

$$\left[\nabla - \left(\frac{2\omega+3}{\omega+2}\right)\frac{\overrightarrow{g}}{c^2}\right] \cdot \overrightarrow{E} = 4\pi\rho\left[1 + \frac{\omega\Phi}{(\omega+2)c^2}\right], \tag{21}$$

$$\nabla \cdot \overrightarrow{B} = 0, \tag{22}$$

$$\nabla \times \overrightarrow{E} = 0, \tag{23}$$

$$\nabla \times \overrightarrow{B} = \frac{2}{c^2}\left(\frac{2\omega+3}{2\omega+4}\right)\nabla \times \left(\overrightarrow{E} \times \overrightarrow{A}\right). \tag{24}$$

In this case, we observe that the mass M influences the value of $\overrightarrow{E}$. Additionally, it can be seen that the product $\overrightarrow{E} \times \overrightarrow{A}$ is source for $\overrightarrow{B}$, so that the emergence of the magnetic field is related to the presence of a static electric field, as well as to the angular momentum $\overrightarrow{j}$ of the central body.

Another case, analogous to what we just commented, is to consider the electric field a first-order term, with $\overrightarrow{E}$ and $\overrightarrow{B}$ static again and $\rho = 0$. Then, immediately can be obtained from (17)-(20) the equations:

$$\nabla \cdot \overrightarrow{E} = \frac{2}{c^2}\left(\frac{2\omega+3}{2\omega+4}\right)\nabla \cdot \left(\overrightarrow{B} \times \overrightarrow{A}\right), \tag{25}$$

$$\nabla \cdot \overrightarrow{B} = 0, \tag{26}$$

$$\nabla \times \overrightarrow{E} = 0, \tag{27}$$

$$\left[\nabla + \left(\frac{2\omega+3}{\omega+2}\right)\frac{\overrightarrow{g}}{c^2}\right] \times \overrightarrow{B} = \frac{4\pi}{c}\overrightarrow{J}\left[1 + \frac{(5\omega+6)\Phi}{(\omega+2)c^2}\right]. \tag{28}$$

It is concluded that M influences the value of $\overrightarrow{B}$, while the product $\overrightarrow{B} \times \overrightarrow{A}$ is source for $\overrightarrow{E}$. Hence, a static magnetic field becomes one of the elements necessary for the generation of the electric field.

5 Conclusion

We obtain the equations of electrodynamics in the scalar-tensor theory of Brans-Dicke, considering the weak field approximation and a background metric representing a central rotating body. These equations present terms with the fields Φ, $\overrightarrow{g}$ and $\overrightarrow{A}$, indicating the participation of gravitational effects in the generation of fields $\overrightarrow{E}$ and $\overrightarrow{B}$.

In turn, the presence of $\overrightarrow{A}$ in the equations is related to the gravitational effects due to the rotation of the central body, known as gravitomagnetic effects [15].

It is interesting to mention that the constant ω appears in the equations for the electric and magnetic fields, expressing the effects of the Brans-Dicke scalar field. When we take the limit $\omega \to \infty$, the equations are reduced to those of general relativity.

We present some particular results, allowed by the field equations, which show the participation of the mass M as well as the angular momentum of the rotating central mass in the determination of $\overrightarrow{E}$ and $\overrightarrow{B}$. Finally, it was found that a static electric field contributes to the generation of the magnetic field and vice versa.

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