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Homotopy Properties in Loop Group

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Abstract

In the present paper, it is reported that a wave equation describing progressive waves on a string under the non-gravitational field has such exact wave-solutions as a circle-circular connecting wave-solution f_k which is presented as a new expression, a distorted-circle-U-shape connecting one $\bar{f}_k$, (where k is a modulus of the Jacobian elliptic functions and $0 \leq k \leq 1$), a circle one f_0, a distorted circle one $\bar{f}_0$, a circular one f_1, and a U-shape one $\bar{f}_1$. Here, it is denoted that each of the two connecting wave-solutions, f_k and $\bar{f}_k$, is one homotopy connecting the continuous mapping pair, f_0 (the circle one) and f_1 (the circular one), or $\bar{f}_0$ (the distorted circle one) and $\bar{f}_1$ (the U-shape one), respectively. Regarding novel properties of "Loop Group", there exists homotopy equivalence due to their contractibility between the circle wave-curve f_0 and the circular one f_1, or between the distorted circle one $\bar{f}_0$ and the U-shape one $\bar{f}_1$. Furthermore, let f_0 be a unit circle and $\mathbf{S}^1$ the set of $|z| = 1$. In general, the continuous mapping f_0 which maps z into z^N makes $\mathbf{S}^1$ be N-fold coverings. Then, $\deg f_0 = N$.

Keywords: Homotopy equivalence, N-fold coverings, Degree of mapping,

1 Introduction

In previous papers [1, 2], when the amplitude of the progressive wave propagating on a string is sufficiently small, we found that there exist various linear waves. In another separate paper [3], we studied such an exact wave-solution as a one-circular solitary one [4], a one-circle one, or a U-shaped one.[5] Regarding these exact wave-solutions, they are stable when propagating in one direction. Besides these exact wave-solutions, we have also found a circle-U-shape connecting wave-solution which is changed to a distorted-circle-U-shape connecting one in this paper and a circle-circular connecting one.[5] We have investigated topology of the exact wave-solutions and have derived "Loop Group".[5] In a recent paper, we have made progress the theory of Loop Group.[6, 7]

In this paper, we introduce the new expression of the circle-circular connecting wave-solution instead of the old expression shown in Ref.[6]. Here. we also introduce factorized representations of the wave-solutions [8] and apply them to redefinition of Loop Group. The purpose of the present paper is to modify the previous theory and to study novel properties of Loop Group. The current study concerning homotopy properties of the continuous mappings makes an understanding of Loop Group advance to a better extent. Moreover, the analytic expression connecting the homotopic continuous mappings will contribute to the field of two dimensional topology so much. Here, we introduce the new expression of the circle-circular connecting wave-solution and the distorted-circle-U-shape connecting one instead of the circle-U-shape connecting one[6]. And, we study a homotopy equivalence. We also introduce degree of the homotopic continuous mappings as N-fold coverings of a unit circle corresponding to N loops in Ref.[5] where N is an integer. And, we redefine Loop Group and emphasize that the two connecting wave solutions for $0 < k < 1$ are no elements of Loop Group. We also refer to the relation that the two fundamental elements f_0 (circle wave-solution) and $\bar{f}_0$ (distorted circle one) can be transformed each other through the related equation.

2 Summary of exact wave-solutions

First, we assume that no external forces such as gravity are exerted on a string (cord) and that stretching and contraction of a string are negligible. And, we let a string lie along the x-axis and a wave propagate on the xy-plane.

The original equation of motion for the string reads[1]

$$\sigma \frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial z}{\partial s}\right), \tag{2.1}$$

where σ is a linear density, s an arclength along the string, and T a norm of a tension vector. Here, we should remark that

$$z = x + iy, \tag{2.2}$$

$$z_s = \mathrm{e}^{i\theta} = \cos\theta + i\sin\theta, \tag{2.3}$$

$$x_s^2 + y_s^2 = 1, \tag{2.4}$$

where $(\cos\theta, \sin\theta)$ is a unit tangential vector, and subscript s denotes partial differentiation with respect to s here and hereafter. In the previous paper,[1] we have proved that T is constant when there exists such a form of a solution as $z(s,t) = z(\xi) = z(s \pm ct)$, where c is a positive constant. When T is constant, Eq. (2.1) reduces to

$$\frac{\partial^2 z}{\partial t^2} - c^2 \frac{\partial^2 z}{\partial s^2} = 0. \tag{2.5}$$

Due to this equation, we find that solutions of this equation have arbitrariness of linear equations with respect to ξ and t.[3] Here, we sum up the result of exact wave-solutions for Eq. (2.5).[5] The below connecting wave-solutions are shown to be the homotopy connecting the continuous mappings such as the circle wave-solution and the circular solitary one, or the distorted circle one and the U-shape one in the following section.

2.1 circle-circular connecting wave-solution f_k and distorted-circle-U-shape connecting one $\bar{f}_k$

We have introduced the old expression of the circle-circular connecting wave-solution in the reference[5]. However, it has the problem that we can show the difference between the result obtained by taking the limit after the integration and the one obtained by the integration after taking the limit. Hence, though introducing the new one in an artificial manner, it is shown as follows.

$$z_1 = -2\int \left(\mathrm{cn}\kappa\xi\mathrm{dn}\kappa\xi - \frac{k-1}{2} + i\mathrm{sn}\kappa\xi\mathrm{dn}\kappa\xi\right)\mathrm{d}\xi + s + k_2(t), \tag{2.6}$$

$$= -2\left[\frac{1}{\kappa}\mathrm{sn}\kappa\xi - \frac{k-1}{2}\xi - \frac{i}{\kappa}\mathrm{cn}\kappa\xi\right] + s + k_2(t) \equiv f_k, \tag{2.7}$$

where $\mathrm{cn}, \mathrm{sn}, \mathrm{and}\,\mathrm{dn}$ are the Jacobian elliptic functions, k is a modulus of the Jacobian elliptic function $(0 \le k \le 1)$, and $\xi = s - s_0 \pm ct$. The distorted-circle-U-shape connecting wave-solution reads [5]

$$z_2 = x + iy = \int (\mathrm{sn}\kappa\xi - 1 + i\mathrm{cn}\kappa\xi)\mathrm{d}\xi + s + k_2(t), \tag{2.8}$$

where $\xi = s - s_0 \pm ct$. Here, define

$$\mathrm{Sn}(u,k) = \int \mathrm{sn}(u,k)\mathrm{du} = \pm(1/k)\log(\mathrm{dn}u \mp k\mathrm{cn}u) + C_1, \tag{2.9}$$

$$\mathrm{Cn}(u,k) = \int \mathrm{cn}(u,k)\mathrm{du} = \pm(1/k)\sin^{-1}(\pm k\mathrm{sn}u) + C_2. \tag{2.10}$$

Combining these equations with Eq. (2.8) yields

$$z_2 = x + iy = (1/\kappa)\mathrm{Sn}\kappa\xi + (i/\kappa)\mathrm{Cn}\kappa\xi + s_0 \mp ct + k_2(t) \equiv \bar{f}_k. \tag{2.11}$$

2.2 circle wave-solution f_0, circular one f_1, distorted-circle one $\bar{f}_0$, U-shape one $\bar{f}_1$

circle wave-solution f_0 reads

$$z = x + iy = -(2/\kappa)\sin(\kappa\xi) + (2i/\kappa)\cos(\kappa\xi) \pm ct \equiv f_0. \tag{2.12}$$

This circle wave equation means that a rotating circle around its center moves leftward or rightward along the x-axis. cf. Fig.1. Next, we show a circular solitary wave-solution in Fig.4. Its parametric representation reads[5]

$$z = x + iy = s - (2/\kappa)\left[\tanh(\kappa\xi) + 1\right] + i(2/\kappa)\mathrm{sech}(\kappa\xi) \equiv f_1, \tag{2.13}$$

Next, the distorted circle wave-solution reads

$$z = x - x_0 \pm ct + iy = (1/\kappa)\mathrm{Sn}(\kappa\xi, 0) + (i/\kappa)\mathrm{Cn}(\kappa\xi, 0) \equiv \bar{f}_0. \tag{2.14}$$

cf. Fig.5. Finally, the U-shape wave-solution reads

$$z = x - x_0 \pm ct + iy = \pm(1/\kappa)\log[\cosh(\kappa\xi)] + i(2/\kappa)\arctan[\exp(\pm\kappa\xi)] \equiv \bar{f}_1. \tag{2.15}$$

We note that the curve moves leftward or rightward by continuing to pull the both ends of the string along the x-axis with a constant velocity c. This is a U-shape wave-solution. cf. Fig.8.

2.3 Two kinds of wave-solutions connecting two pairs of ones

Here, we point out the problem that the result obtained by the integration in Eq.(2.8) after taking the limit $k \to 0$ is different from the one obtained by the limit $k \to 0$ in Eq.(2.11) after integrating. In this subsection, we consider the case taking the limits after integrating. When $k = 1$, the two connecting wave-solutions f_k and $\bar{f}_k$ become the correct circular and U-shape ones f_1 and $\bar{f}_1$, respectively (cf, Figs.4,8). We investigate this fact by showing figures of the two connecting wave solutions for various values of k in Figs.4-13. As shown there, we find that the distorted-circle-U-shape connecting wave one becomes no circle but a distorted circle when $k \to 0$ (cf. Fig.5). Finally, we verify that each of the circle-circular connecting wave-solution f_k and the distorted-circle-U-shape connecting one $\bar{f}_k$ is one homotopy connecting the continuous mapping pair f_0(circle wave) and f_1(circular one), or $\bar{f}_0$(distorted circle one) and $\bar{f}_1$(U-shape one), respectively, where k is a modulus of the Jacobian elliptic functions and $0 \leq k \leq 1$.

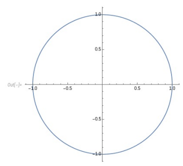

Figure 1: Circle-circular wave when $k = 0.0$

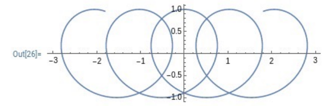

Figure 2: when $k = 0.3$.

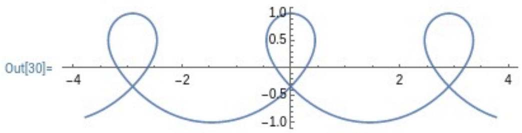

Figure 3: when $k = 0.7$

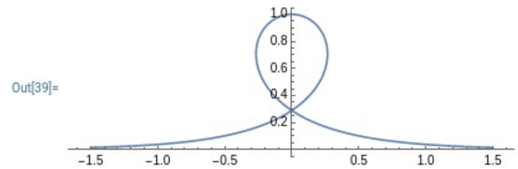

Figure 4: when $k = 1.0$.

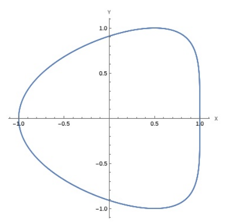

Figure 5: Distorted-circle-U-shape wave when $k = 0.01$.

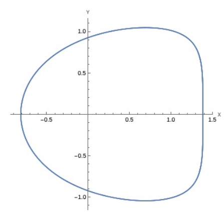

Figure 6: when $k = 0.5$

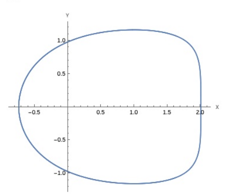

Figure 7: when $k = 0.8$.

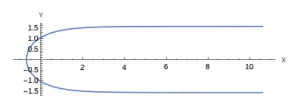

Figure 8: when $k = 1.0$.

3 Factorized representations of the exact wave-solutions

Here, we deal with the wave problem under the tangential angle θ frame. First, we derive the wave equation for θ by taking the derivatives with respect to s for x and y in Eq. (2.5), then, we have

$$(\theta_t - c\theta_s)(\theta_t + c\theta_s) = 0, \tag{3.1}$$

where

$$\exp(i\theta) = \cos\theta + i\sin\theta = \partial x/\partial s + i\partial y/\partial s, \tag{3.2}$$

is used. Equation (3.1) has a general solution which describes the propagation of waves in a plane,

$$\theta(s,t) = f(s-ct) \quad \text{or} \quad f(s+ct), \tag{3.3}$$

where f is an arbitrary function. By the superposition principle, it is proved that a linear combination of 1-solitary wave-solutions with only the same velocities satisfies Eq. (3.1).

When we cite Ref.[8], the factorized form of the N-circular solitary wave-solution reads

$$\exp(i\theta) = \cos\theta + i\sin\theta = \prod_{m=1}^{N} \frac{\sinh A_m - i}{\sinh A_m + i}, \tag{3.4}$$

$$A_m = \kappa(s - s_m - ct). \tag{3.5}$$

From Eq. (2.15) and Eq. (3.2), we have for the N-U-shape solitary wave-solution

$$\exp(i\theta) = \cos\theta + i\sin\theta = \prod_{m=1}^{N} \left[\frac{\sinh A_m - i}{\sinh A_m + i}\right]^{1/2}. \tag{3.6}$$

From Eq. (2.12) and Eq. (3.2), we have for the N-circle solitary wave-solution

$$\exp(i\theta) = -2\cos\theta - 2i\sin\theta = \prod_{m=1}^{N} [-2\exp(iA_m)]. \tag{3.7}$$

Here, we show the factorized forms of the two kinds of the N-connecting wave-solutions in the same manner as in the previous sunsection. First for the circle-circular connecting wave-solution, taking Eq. (2.6) and Eq. (3.2) into account, we have

$$\exp(i\theta) = \cos\theta + i\sin\theta = \prod_{m=1}^{N} \left[-2(\mathrm{cn}A_m \mathrm{dn}A_m - \frac{k}{2} + i\mathrm{sn}A_m \mathrm{dn}A_m)\right]. \tag{3.8}$$

Next, regarding the distorted-circle-U-shape connecting wave-solution, from Eq. (2.8) and Eq. (3.2), we obtain

$$\exp(i\theta) = \cos\theta + i\sin\theta = \prod_{m=1}^{N} [\mathrm{sn}A_m + i\mathrm{cn}A_m] \tag{3.9}$$

4 Homotopy properties in loop group

4.1 Homotopy equivalence

We consider a set of various wave-curves in a two dimensional topological space. We assume that both ends of a string in the U-shape wave-curve connect themselves at a infinitely far point, then we obtain a circle-like one because it is denoted to be contractible. In the same manner, when both ends of a string in the circular solitary one approach each other and connect themselves finally, we have a circle one due to contractibility to be satisfied. In such a manner, the four kinds of solitary ones are homotopy equivalence. On the contrary to the contractibility, we consider to stretch the two ends, which are made by cutting the circle wave-curve at one point, to infinitely far point(s). If we stretch them to the same direction along the x-axis, we have the U-shape one. If we stretch them to the opposite directions along the x-axis, we obtain the circular solitary one. Therefore, we predict that there will exist nothing but only these four wave-curves that are homotopy equivalence.

4.2 Degree of continuous mappings

Let $\mathbf{S}^1$ be a set of a unit circle, namely, $\mathbf{S}^1 = \{z \mid |z| = 1\}$, and f be a continuous mapping $f : \mathbf{S}^1 \to \mathbf{S}^1$ and the anticlockwise direction be positive. Then, f which maps z into z^N ($f(z) = z^N$) yields N-fold coverings on $\mathbf{S}^1$ and degree of the continuous mapping f is $\deg f = N$. In the current problem, we can let the circle wave-solution f_0 (cf. Eq.(2.12)) be a unit circle ($t = 0$, or on a moving coordinate system) without loss of generality. Then, $\deg f_0 = 1 (N = 1)$. When continuous mappings f, g are homotopic, $\deg f = \deg g$ holds. Accordingly, regarding the circle f_0 and the circular f_1, or the distorted circle $\bar{f}_0$ and the U-shape curve $\bar{f}_1$, the relation $\deg f_0 = \deg f_1 = 1$, or $\deg \bar{f}_0 = \deg \bar{f}_1 = 1$ holds since they are homotopic. In the factorized form of the N-circle solitary wave-solution (cf. Eq. (3.7)), we assume that $A_1 = A_2 = ... = A_N$, then, f_0 which maps z into z^N yields N-fold coverings on $\mathbf{S}^1$ and degree of the continuous mapping f_0 is $\deg f_0 = N$.

4.3 Redefinition of loop group

We have considered that an element of the loop group has N-loops and the solitary wave-solutions as fundamental elements can be transformed each other. In other words, we can interpret it as the operation of the two connecting wave-solutions. From this viewpoint, we shall investigate again the definition of the loop group.

Regarding the current homotopy f_k connecting the continuous mapping pair f_0 and f_1, or the current homotopy $\bar{f}_k$ connecting the continuous mapping pair $\bar{f}_0$ and $\bar{f}_1$, it will yield 1-fold covering on a unit circle. However, as a general case of N-solitary wave-solutions, a homotopy g_k, or $\bar{g}_k$ which yields N-fold coverings on a unit circle has remained unknown yet. Hence, we exclude the two connecting homotopy f_k and $\bar{f}_k$ for $0 < k < 1$ from the elements of Loop Group. They only behave so as to connect each of the two continuous mapping pairs, f_0 and f_1, or $\bar{f}_0$ and $\bar{f}_1$. And, we have a question whether there exists the way how the distorted circle wave-solution $\bar{f}_0$ and the circle one f_0 are connected, or not. However, we refer to the relations between them and between the two connecting wave-solutions which are homotopy. Accordingly, they will be connected each other through the analytic expressions. To clarify what it means as Loop Group is our future task.

Due to the factorized representations, we redefine the operation of elements a, b in loop group to be $a \circ b = a \cdot b$, the inverse element a^{-1} of a to be $1/a$, and the unit element to be $a \cdot (1/a) = 1$. Then, we have "loop group" without inconsistency.

5 Discussion

Being different from the knot theory, in the above-mentioned manner, two kinds of the conncting wave-solutions f_k and $\bar{f}_k$ can realize not manually but analytically the transormations of the string shapes between two continuous mapping pairs, f_0 and f_1, and $\bar{f}_0$ and $\bar{f}_1$ which are the fundamental elements of Loop Group in the two dimensional topology. And, to verify the conjecture that Loop Group will be a fundamental group with a fixed base point is our future task.

At the end of this paper, we present a question whether there exists nothing but only the continuous mapping pairs, f_0 and f_1, and $\bar{f}_0$ and $\bar{f}_1$ in Loop Group, or not. This question is our future subject to be certified.

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References

[1] T. Shimizu and K. Nakayama, Wave equation for cord and its various wave solutions under non-gravitational field, *Adv. Stud. Theor. Phys.*, **13** (2019), no. 8, 387-399. https://doi.org/10.12988/astp.2019.9936

[2] T. Shimizu and K. Nakayama, Various Approximate Solutions in Wave Equation for Cord under non-gravitational field, *Adv. Stud. Theor. Phys.*, **13** (2019), no. 8, 401-412. https://doi.org/10.12988/astp.2019.91038

[3] K. Nakayama and T. Shimizu, Various exact solutions in wave equation for cord under non-gravitational field. (to be published).

[4] T. Shimizu, K. Sawada and M. Wadati, Determination of the one-kink curve of an elastic wire through the inverse method, *J. Phys. Soc. Jpn.*, **52** (1983), 36-43. https://doi.org/10.1143/jpsj.52.36

[5] T. Shimizu and K. Nakayama, Five Kinds of Exact Solutions in Wave Equation for String and "Loop Group", *Adv. Stud. Theor. Phys.*, **13** (2019), no. 8, 413-424. https://doi.org/10.12988/astp.2019.91141

[6] T. Shimizu, Loop Group and Homotopy Class as Additive Group, *Adv. Stud. Theor. Phys.*, **15** (2021), no. 8, 373-381. https://doi.org/10.12988/astp.2021.91708

[7] Stefan Schwede, *Global Homotopy Theory*, Cambridge University Press, Cambridge, 2018.

[8] T. Shimizu, Three-Dimensional Wave Equation for a Cord and N-Spiral Solitary wave solution, *J. Phys. A: Mathematical and General*, **27** (1994), 4601-4609. https://doi.org/10.1088/0305-4470/27/13/033

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