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From Vacuum to Dark Energy. Exact Anisotropic Cosmological Solution of Petrov Type D for a Nonlinear Scalar Field

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Abstract

Two exact solutions to Einstein equations, which differ because of its type of initial expansion, are obtained to a nonlinear scalar field with a potential type $V = \Lambda\left(1 - tanh(\frac{\sqrt{6}\phi}{2})^4\right)$. It is determined that the energy density of solutions is not singular for any time value and for which at the beginning in $t = 0$, the space-time is a vacuum of Kasner type ($a_1 = a_2 = -2a_3 = 2/3$) for one solution and the flat world for the other. By having studied the temperature, it is established that it is null at the beginning and that once it increases up to a maximum value, it stops increases and asymptotically goes down to zero in respect to time. The Hubble and deceleration parameters were studied, it is showed that the Hubble parameter is indefinite in $t = 0$ and tends to have a constant value as time increases; then, the deceleration parameter indicates an initial process of a decelerated expansion that continuously changes into an accelerated one as time increases. By the study of the Jacobi stability of the solutions, it is obtained that the solutions are initially unstability but cease to be so in a determined time. The space-time of both solutions transforms into the equivalent of dark energy for FRWL as time increases.

Keywords: cosmology, exact solution, Einstein, scalar field, dark energy, temperature, non-lineal, Kretschmann, Hubble, deceleration

1 Introduction

Currently, Cosmology is a topic of great interest. Since the beginning of the century and thanks to the data obtained by COBE, WMAP, and PLANCK satellites as well as the discovery of the universe acceleration, some concerns have arrived and have been discussed all together with other aspects and its respective literature in [1]. On the other hand, some recent works state that considering the universe as isotropic and homogeneous may be an issue [2]. Moreover, some findings from the James Webb telescope, please see [3], might help to promote the interest towards new cosmological models.
The study of scalar fields and the Jacobi stability of solutions in a homogeneous anisotropic symmetry of the Petrov D type has been discussed in [4], as well as other aspects of interest and their respective literature. The present work studies a nonlinear scalar field that represents the type of substance necessary for a universe to begin from a vacuum and, continuously, to be transform into dark energy. The stability of the solutions is studied, as well as the parameters of Hubble, deceleration and temperature.

2 Symmetry and the Nonlinear Scalar Field

2.1 The Anisotropic Symmetry of Space-time of Petrov Type D and the Einstein Tensor

The symmetry used in this work is anisotropic and homogeneous of Petrov type D and has the form [1]

$$ds^2 = Fdt^2 - t^{2/3}K(dx^2 + dy^2) - \frac{t^{2/3}}{K^2}dz^2, \tag{1}$$

where F and K are functions of t.
Einstein tensor components ($G_\alpha^\beta = R_\alpha^\beta - 1/2\delta_\alpha^\beta R$) different from zero, of (1), are

$$G_0^0 = \frac{4\,K^2 - 9\,t^2\dot{K}^2}{12t^2K^2F}, \tag{2}$$

$$G_1^1 = -\frac{3Kt\dot{K}\left(2F - \dot{F}t\right) + 3Ft^2\left(2K\ddot{K} - 5\dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{3}$$

$$G_2^2 = G_1^1 = -\frac{G_3^3}{2} + \frac{9\,Ft^2\dot{K}^2 - 4\,K^2\dot{F}\,t - 4\,K^2F}{8t^2K^2F^2}, \tag{4}$$

where the points on the functions represent derivatives by time.

2.2 The Scalar Field

The scalar field of this work is determined by the following Lagrangian:

$$L_{sc} = \frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} - V(\phi) \tag{5}$$

where it is considered that $V(\phi) = \Lambda\left(1 - tanh(\frac{\sqrt{6}\phi}{2})^4\right)$ and where $\Lambda > 0$. This represents the scalar potential of interaction and ϕ, the function of the scalar field.
By noticing that due to the spatial homogeneity (see G^{μ}_{μ}), the only components of the stressenergy tensor different from zero are T^0_0, $T^1_1 = T^2_2$ and T^3_3. It is stated that the scalar function $\phi = \phi(t)$ and that the stressenergy tensor of (5) has the form

$$T^{\beta}_{\alpha} = \frac{\dot{\phi}^2}{F}\delta^{\beta}_0\delta^0_{\alpha} - \delta^{\beta}_{\alpha}L_{sc} \tag{6}$$

where the point on ϕ represents the partial derivative by t.

2.3 Einstein Equations and its Solutions

Einstein equations have the form $G^{\beta}_{\alpha} = \kappa T^{\beta}_{\alpha}$ (considering for its convenience $\kappa = 1$). From (2-4 and 6), the following system of equations independent of each other is obtained

$$G^0_0 - \frac{\dot{\phi}^2 + 2V(\phi)F}{2F} = 0, \tag{7}$$

$$G^1_1 - \frac{-\dot{\phi}^2 + 2V(\phi)F}{2F} = 0, \tag{8}$$

$$G^3_3 - \frac{-\dot{\phi}^2 + 2V(\phi)F}{2F} = 0, \tag{9}$$

By subtracting the equality (8) with (9), the result is

$$\dot{K}K\left(2F - \dot{F}t\right) - 2Ft\left(-K\ddot{K} + \dot{K}^2\right) = 0 \tag{10}$$

where

$$K = K_0 e^{C_1\int\frac{F^{1/2}}{t}dt} \tag{11}$$

Without loss of generalities, it is possible to take it in (11) to $K_0 = 1$ and the constant $C_1 = \pm 2/3$ what offers two possible situations as it can be seen

below.
Einstein equations (7, 8 and 9) when considering (11) take the form

$$-\frac{1}{3}\frac{F-1}{Ft^2}-\frac{\dot{\phi}^2+2V(\phi)F}{2F}=0, \tag{12}$$

$$\frac{1}{3}\frac{-\dot{F}t+F^2-F}{F^2t^2}-\frac{-\dot{\phi}^2+2V(\phi)F}{2F}=0. \tag{13}$$

The scalar field equation gets the form

$$V_{,\phi}-\frac{\dot{F}\dot{\phi}t-2\dot{\phi}F-2Ft\ddot{\phi}}{2F^2t}=0, \tag{14}$$

which can also be obtained from $T^{\nu}_{\mu;\nu}=0$ in which $V_{,\phi}$ is the derivative of V by ϕ. Volumetric energy density of the scalar field and pressure are given by

$$\mu=\frac{\dot{\phi}^2+2VF}{2F}, \quad P=\frac{\dot{\phi}^2-2VF}{2F}. \tag{15}$$

By adding (12)+(13), it is obtained that

$$V(\phi)=\frac{-\dot{F}}{6F^2t} \tag{16}$$

and by subtracting (12)-(13)

$$2F^2-2F+3Ft^2\dot{\phi}^2-\dot{F}t=0. \tag{17}$$

By considering in (16) that $V(\phi)=\Lambda\left(1-tanh(\frac{\sqrt{6}\phi}{2})^4\right)$, it is established that

$$\phi(t)=\pm i\sqrt{\frac{2}{3}}\arctan\left(\left(\frac{\dot{F}}{6t\Lambda F^2}+1\right)^{1/4}\right), \tag{18}$$

or

$$\phi(t)=\pm\sqrt{\frac{2}{3}}\,arctanh\left(\left(\frac{\dot{F}}{6t\Lambda F^2}+1\right)^{1/4}\right), \tag{19}$$

The analysis of this case will take place when $\phi(t)$ is given by (18).
By placing ϕ from (18) in (17), it shows that

$$2F^2-2F-\frac{2Ft^2\dot{G}^2}{(1+G^2)^2}-\dot{F}t=0, \; where\; G=(\frac{\dot{F}}{6\Lambda tF^2}+1)^{1/4}. \tag{20}$$

The solution of (20) has to be one different from the solution of $\dot{F}+6\Lambda F^2 t=0$ and of $\sqrt{6}\Lambda t F+\sqrt{(\dot{F}+6\Lambda F^2 t)\Lambda t}=0$. The equation (20) can be also obtained from the equation of the field (14). The solution of interest, in this work, of the equation (20) is

$$F=\frac{1+12t^2\Lambda}{(1+6t^2\Lambda)^2}. \tag{21}$$

When placing (21) in (18), the result is

$$\phi(t)=\pm i\sqrt{\frac{2}{3}}\arctan\left(\frac{1}{\sqrt{1+12t^2\Lambda}}\right) \tag{22}$$

and of (22) in $V(\phi)=\Lambda\left(1-tanh(\frac{\sqrt{6}\phi}{2})^4\right)$,

$$V(\phi(t))=V(t)=\Lambda\left(1-\frac{1}{(1+12t^2\Lambda)^2}\right). \tag{23}$$

The K function of (11) is equal to $K=e^{\pm 2\sigma/3}$ by considering that (21), σ is

$$\sigma=-arctan\left(\frac{1}{\sqrt{1+12t^2\Lambda}}\right)-arctanh\left(\frac{1}{\sqrt{1+12t^2\Lambda}}\right). \tag{24}$$

2.4 Analysis of Solutions

2.4.1 Singularities and Tendencies

For the study of the singularity when $t\to 0$, the Kretschman invariant is used ($Krest=R^{\alpha\beta\gamma\sigma}R_{\alpha\beta\gamma\sigma}$). For the used symmetry and by considering (11), two possible invariants are obtained in relation to the positive or negative value of $C_1=\pm 2/3$ (see [4]) and which take into account (21) have the form

$$\begin{aligned} Krets_{\pm}=\frac{64}{3}\,\frac{\Lambda^2\,(585\,t^4\Lambda^2+2808\,t^6\Lambda^3+1+2592\,t^8\Lambda^4+36\,t^2\Lambda)}{(1+12\,t^2\Lambda)^4}+ \\ +\frac{32}{27\,t^4}\left(1\pm\frac{6\,t^2\Lambda+1}{\sqrt{1+12\,t^2\Lambda}}\right). \end{aligned} \tag{25}$$

From (25), it is established that when $t\to 0$,

$$Krest_{+}\approx\frac{64}{27t^4} \tag{26}$$

and for $Krets_{-}$, the result demonstrates that

$$Krest_{-}\approx 0. \tag{27}$$

The solution with a positive sign in (11) is singular when $t \to 0$; this is a singularity of the solution of the Kasner vacuum E_{D_1} (see ([1])), and with a negative sign, the solution is not singular because it represents the flat world. What has been said is denoted in the metric (1) by placing the respective functions when $t \to 0$, of $F \to 1$ and $K \to e^{\mp 2\pi/3}(3\Lambda)^{\pm 1/3} t^{\pm 2/3}$ and by changing $e^{\mp\pi/3}(3\Lambda)^{\pm 1/6} x = x'$, $e^{\mp\pi/3}(3\Lambda)^{\pm 1/6} y = y'$ and $e^{\pm 2\pi/3}(3\Lambda)^{\mp 1/3} z = z'$, the metric (1) tends to

$$ds^2 \to dt^2 - t^{2/3 \pm 2/3}(dx'^2 + dy'^2) - t^{2/3 \mp 4/3} dz'^2, \tag{28}$$

The solutions of the metric (1) not only tend to a solution of the Kasner vacuum and of the flat world, but also are the Kasner vacuum or the flat world. Because $T^\mu_\nu = 0$ when $t = 0$; then, the Riemann curvature tensor, when $t \to 0$, is not null if it takes K_+, and agrees with the Kasner solution analog E_{D_1}, but is null in t=0 when is considered K_- .
The Kretschmann invariant of the two solutions tends to $Krets_\pm \to 8\Lambda^2/3$ when $t \to \infty$ similarly to the dark energy model (see [1]). This can also be appreciated in (15) since

$$\mu = \frac{12\Lambda^2 t^2}{(1 + 12\Lambda t^2)}, \quad P = \frac{-36\Lambda^2 t^2 (1 + 4\Lambda t^2)}{(1 + 12\Lambda t^2)^2}, \tag{29}$$

that tends to $\mu \to \Lambda$ and $P \to -\Lambda$ when $t \to \infty$, so Λ represents dark energy.

2.4.2 Hubble Parameters and Deceleration

The Hubble parameters H and deceleration q have been studied and used previously (for example [4]). In this work, they are defined and equal to

$$H = \frac{\left((g_{11} g_{22} g_{33})^{1/6}\right)^{\cdot}}{\sqrt{g_{00}}(g_{11} g_{22} g_{33})^{1/6}} = \frac{6\, t^2 \Lambda + 1}{3\, t \sqrt{1 + 12\, t^2 \Lambda}} \tag{30}$$

and

$$q = -(1 + \frac{\dot{H}}{\sqrt{g_{00}} H^2}) = 2 \frac{18\, t^2 \Lambda + 1 - 36\, t^4 \Lambda^2}{(1 + 12\, t^2 \Lambda)(1 + 6\, t^2 \Lambda)}. \tag{31}$$

When $t \to 0$, Hubble parameters and deceleration tend to $H \to \infty$ and $q \to 2$, and when $t \to \infty$, they tend to $H \to \sqrt{\Lambda/3}$ and $q \to -1$ which indicates that a change in the process of expansion of the universe exists and which initially decelerates and then, accelerates passing through a moment where it does not accelerate either decelerate in $t_{q=0} = \frac{\sqrt{3}(12+4\sqrt{13})}{24\sqrt{\Lambda(3+\sqrt{13})}}$.

3 Temperature

The temperature for the fluid type studied is obtained as in [5], so

$$\frac{dP}{\mu + P} = \frac{dT}{T}, \tag{32}$$

where T is the fluid temperature. From the solution (29) for μ and P, and of (32), it is obtained that temperature depends on the time form

$$T = \frac{128\,\Lambda^{3/2} T_{max} t^3}{\left(1 + 12\,t^2 \Lambda\right)^2}, \tag{33}$$

where $T_{max} > 0$ is an integration constant which represents the maximum temperature. From (33), it is stated that when $t \to \infty$ and when $t \to 0$, temperature tends to zero. The first limit is due to the scalar field tendency to behave as a dark-energy fluid for high values of t, and the second limit because it has, in the beginning, a vacuum of Kasner type E_{D_1} or a flat space-time. Temperature has a maximum value in $t_{T_{max}} = (2\sqrt{\Lambda})^{-1}$. Accordingly, a universe that is cold in its beginning $T_{start} = 0$ gets heated up to a maximum value of $T = T_{max}$ in $t_{T_{max}}$, and it turns cold again asymptotically as time increases $T \approx (8T_{max})/(9\sqrt{\Lambda}t) \to 0$.

4 Jacobi Stability

The Jacobi stability in solutions, where the symmetry (1) has been used, has been studied in [4] and it is based on [6] where more details about this procedure can be found. In this work, the key elements from [4] will be used in order to establish the Jacobi stability in the solution.

As per the order in the Kosambi-Cartan-Chern theory (KCC) [6], it will be defined $x^1 = F$, $y^1 = \dot{F}$, $x^2 = \phi$, $y^2 = \dot{\phi}$, so the local coefficients of the semispray $G^1 = G^1\left(x^1, x^2, y^1, y^2\right)$ and $G^2 = G^2\left(x^1, x^2, y^1, y^2\right)$ take the form

$$G^1 = -\frac{V' y^2 y^1}{2V} + 3V(x^1)^2 - \frac{(y^1)^2}{x^1}, \quad G^2 = \frac{x^1 V'}{2} - \frac{y^2 y^1}{4x^1} - \frac{3(x^1)^2 V y^2}{y^1}, \tag{34}$$

where V' is the partial derivative of V by x^2.

For the study of the Jacobi stability or the KCC theory and according to Routh-Hurwitz's criterion, it is established that if

$$Neg = P_1^1 + P_2^2 < 0, \quad Pos = P_1^1 P_2^2 - P_2^1 P_1^2 > 0, \tag{35}$$

the system is Jacobi stable and on the contrary it is not, where P^i_j is the second KCC-invariant or deviation tensor of the curvature which can be defined as

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j, \tag{36}$$

in turn, the nonlinear coefficients of connection N^i_j and the Berwald connection G^i_{jl} are defined in the form

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{37}$$

The relations Neg and Pos in (35) by taking into consideration the solutions (21), (16) and by obtaining $\dot{\phi}$ from (22) have the form

$$Neg = -\frac{3(432t^4\Lambda^2 + 98t^2\Lambda - 1)}{4t^2(1+12t^2\Lambda)^2(1+6t^2\Lambda)} \tag{38}$$

and

$$Pos = \frac{36\Lambda(1728t^6\Lambda^3 - 552t^4\Lambda^2 - 162t^2\Lambda - 7)}{t^2(1+12t^2\Lambda)^4(1+6t^2\Lambda)^2}, \tag{39}$$

from (38) and (39) and when considering (35), as a result, the solution is Jacobi stable for any value of

$$|t| > \frac{\sqrt{3}(36+4\sqrt{249})}{48\sqrt{\Lambda(9+\sqrt{249})}}. \tag{40}$$

The absolute value of time $|t|$ in (40) has been considered, so if the symmetry in (1) is changed to

$$ds^2 = Fdt^2 - (t^2)^{1/3}K(dx^2+dy^2) - \frac{(t^2)^{1/3}}{K^2}dz^2, \tag{41}$$

if the change of $t \to -t$ does not take place, changes in the whole solution will not be present, so it ends up being symmetric in respect to time. An interpretation of this type of solution has been discussed in [7].

From the study of the Jacobi stability, it is established that the solution is not Jacobi stable for times that do not meet (40), so the universe goes through a no-stable beginning ("chaotic") and after a period of time, it turns stable. In regards to the analyzed case within this work, what has been stated previously can be interpreted by considering the scalar field as a fluid in such a way that:

1. Changes in F so $u_0 = \sqrt{F}$ that is the component 0 of the four-dimensional covariant velocity vector and for which is supposed that $u_\nu = \{u_0, u_1, u_2, u_3\} = \{\sqrt{F}, 0, 0, 0\}$. It distances from this state causing motion in u_ν as the stability assumes that with a little motion it should come back to its original state, but during certain interval, this does not happen (although, it becomes stable afterwards). It means that in time when it is not stable, it can lead to relative

velocities in parts of the flux which were initially very close and which possibly create chaos by means of an "attractor's type" or a deterministic chaos. 2. The primary instability of the field ϕ implies that the field initially represents a type of scalar matter that starts "filling up" the vacuum and as time (absolute value of time) increases its material sense changes. In other words, in any determined time, similar substances (or particles) can change or transform differently with time until they turn stable and keeping their previously acquired differences. A similar situation, although less satisfactory from the Jacobi stability point of view, happens for a Higgs-like field $V(\phi) = V_0 + M^2\phi^2/2 + \lambda\phi^4/4$ in a FRWL symmetry and for which the stability is met, in the best-case scenario, in different time intervals interspersedly [6] in which the stability concentrates in the scalar field and in the "volume" (scalar factor). In the case of symmetry (1) and the scalar potential $V(\phi) = \Lambda\left(1 - tanh(\frac{\sqrt{6}\phi}{2})^4\right)$ with the increase of $t \to \infty$, $\phi \to 0$ and $V(\phi) \to \Lambda - 9\Lambda\phi^4/4$ and similarly to the Higgs-type field, it is a potential of quartic interaction that turns out stable.

5 Conclusions

In this work, two exact solutions from a nonlinear scalar field with a potential equal to $V(\phi) = \Lambda\left(1 - tanh(\frac{\sqrt{6}\phi}{2})^4\right)$ were obtained and represent cosmological models very close to each other for high-time intervals and different in relation to the form in which the universe expands and to what they initially represent. In both cases, the universe becomes equivalent to dark energy as time increases but when it is close to $t = 0$, one of them goes through a singularity, a Kasner vacuum E_{D_1} [1]. This generates a greater expansion in the axis z. The other solution nearby $t = 0$ (at that point) generates a greater expansion in the plane xy, and the space-time is flat in $t = 0$; in both cases, the stress-energy tensor is null in $t = 0$, so vacuum appears in the models. After studying the Jacobi stability of the solutions, the result showed that they were Jacobi stable from a preset time value in (40). Previously, from $|t| = 0$, it is Jacobi unstable which implies a chaotic start but very active in relation to the increasing volumetric density of the field μ. As a result of the study of Hubble parameters and deceleration, the Hubble parameter is indefinite in $t = 0$ but asymptotically tends to a constant value when t increases and as $H \to \sqrt{\Lambda/3}$ does it. Meanwhile, the deceleration parameter initially tends to $q \to 2$ and for higher time intervals, it tends to $q \to -1$ which represents that in both models the universe initially decelerates up to a time lower than $t_{q=0} = \frac{\sqrt{3}(12+4\sqrt{13})}{24\sqrt{\Lambda(3+\sqrt{13})}}$; as of this time, the universe starts an asymptotic accelerated expansion process to a constant value of $q = 1$. In the study of the scalar field temperature in which the field was considered a fluid, in both models, the universe is cold in the beginning; it is vacuum. Then, it increases its temperature until it gets a maximum value in a time equal to $t_{T_{max}} = (2\sqrt{\Lambda})^{-1}$ after which it decreases its value to zero asymptotically. The solutions have temporal symmetry if the metric is given by (41).

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