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Anisotropic Cosmological Exact Solutions of Petrov Type D of a Mixture of Dark Energy and an Attractive Bose-Einstein Condensate

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Abstract

Two exact solutions to Einstein equations were obtained; whose difference is the kind of initial expansion (greater on an axis or on a plane) in an anisotropic and homogeneous symmetry of Petrov type D. In these solutions, a mixture of dark energy and a "standard" attractive Bose-Einstein Condensate (BEC) were considered. The initial singularity problem was studied and is only present in one of these models. Also, the study of temperature allowed to determine that it is null at the start of the universe and as time increases asymptotically, it tends to a constant value, hence a BEC is essential in the first expansion phase and, for this work, with 8/9 parts of the total early universe energy. Hubble and deceleration parameters were studied as well and it was determined that the Hubble parameter initially gets indefinite in $t = 0$, but tends to a constant value when time increases; for the deceleration parameter, it was stated that the universe initially expands in a decelerated form, but from a certain value, it starts continuously to expand fast towards a constant value. As time increases asymptotically, models tend to a one of FRWL flat type of dark energy.

Keywords: cosmology, exact solution, Einstein, temperature, Hubble, deceleration parameter, Kretschmann, singularity, Bose-Einstein condensate, dark energy

1 Introduction

Interest in Cosmology has redirected the attention to other possible models apart from the standard flat type ΛCDM due to new observations of the universe; these have been discussed in [1] where their respective literature can be found. Besides those studies, new ones have increased the interest for alternative cosmological models such as: the study from the Zhu & Ménard's [2] catalogs that determine the presence of "a giant arc" and data from the James Webb Space Telescope [3] related to high-redshift galaxies.

A "standard" attractive Bose-Einstein Condensate (BEC) can be represented by means of an equation of state (EoS) of a nonlinear fluid (polytropic index of n=1) of type $P_{BEC} = -a\mu_{BEC}^2$ where P_{BEC}, μ_{BEC} and a are the pressure, the volumetric energy density, and a positive constant of BEC's self-interaction [4]. This kind of EoS has been considered a dark matter model as in [4]. Moreover, a BEC is present in low temperature which implies a high thermodynamic condition to the fluid for a cosmological model; this could be essentially present in the early phases of the universe as it is the case in this work. Furthermore, a universe model must have dark energy as per what has been seen and which infers that it could be a mixture of a BEC and dark energy. Interestingly, it has been established in an experimental manner that the dynamic of a ring-shaped BEC that expands supersonically has a redshift from phonons similar to the shift of photons of the universe [5] and conduces to the production of topological excitations. Additionally, the slowing-down of light to almost stop it took place by means of a BEC in 1999 that created an induced quantum interference that allowed light to have a speed of 17 m/s in the laboratory [6]. That is to say if the universe in its early phase behaved partly as a BEC then, it can be assumed hypothetically that some conditions could have appeared, in a region, that allowed light to slow down fast its speed, so some observable processes from these conditions could not coincide with given time estimations of these phenomena, could not be detected nor could they provide any correct information of what took place. In this work, two solutions are obtained; they differ from each other because of their initial expansion either by one of their axes or plane which provides a favorable condition by considering the previous hypothesis where light slows down its speed and forms anisotropies of greater importance to a spatial axis.

2 Symmetry, Einstein Equations and Solutions

The symmetry used in this work is anisotropic and homogeneous of Petrov type D and has the form 1

$$ds^2 = Fdt^2 - t^{2/3}K(dx^2 + dy^2) - \frac{t^{2/3}}{K^2}dz^2, \tag{1}$$

where F and K are functions of t.
Einstein tensor components ($G_{\alpha}^{\beta} = R_{\alpha}^{\beta} - \frac{1}{2}\delta_{\alpha}^{\beta}R$) different from zero, of (1), are

$$G_0^0 = \frac{4\,K^2 - 9\,t^2\dot{K}^2}{12t^2K^2F}, \tag{2}$$

$$G_1^1 = -\frac{3Kt\dot{K}\left(2F - \dot{F}t\right) + 3Ft^2\left(2K\ddot{K} - 5\dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{3}$$

$$G_2^2 = G_1^1 = -\frac{G_3^3}{2} + \frac{9\,Ft^2\dot{K}^2 - 4\,K^2\dot{F}\,t - 4\,K^2F}{8t^2K^2F^2}, \tag{4}$$

where the points on the functions represent derivatives by time.

The perfect fluid model used in Cosmology (as the one of dark energy or of a BEC) represents a nonviscous and isentropic fluid ($P = P\,(\mu)$) without shear stress and which can be expressed as

$$T_{\alpha\beta} = (\mu_T + P_T)\,u_\alpha u_\beta - g_{\alpha\beta}P_T, \tag{5}$$

where $T_{\alpha\beta}$ is the stressenergy tensor of a fluid mixture with minimum interaction, u_α is the tetradimensional speed, $g_{\alpha\beta}$ is the metric tensor, μ_T and P_T are the total energy density and the total pressure of the fluid mixture respectively.

The equation of state for the mixture of an attractive BEC and dark energy will be taken from the form

$$P_T = P_{DE} + P_{BEC} = -\mu_{DE} - a\mu_{BEC}^2,\ \mathit{where}\, a > 0 \tag{6}$$

where $\mu_{DE} = \Lambda$ and μ_{BEC} represent the energy density of dark energy and BEC fluids respectively and $P_{DE} = -\Lambda$ and $P_{BEC} = -a\mu_{BEC}^2$, their respective pressure.
A fluid with a tetradimensional speed will be considered $u_\alpha = (u_0, 0, 0, 0)$, so the components of the stressenergy tensor (5) different from zero are $T_0^0 = \Lambda + \mu_{BEC}$, $T_1^1 = T_2^2 = T_3^3 = -\Lambda - P_{BEC}$, which implies that Einstein equations $G_\alpha^\beta = \kappa T_\alpha^\beta$ have to fulfill that $G_1^1 = G_3^3$, so from (3) and (4), it is obtained

$$\dot{K}K\left(2F - \dot{F}t\right) - 2Ft\left(-K\ddot{K} + \dot{K}^2\right) = 0 \tag{7}$$

thus,

$$K = K_0 e^{C_1\,\sigma},\ \mathit{where}\ \sigma := \int \frac{\sqrt{F}}{t}dt, \tag{8}$$

without loss of generality, the constant K_0 in (8) will be considered equal to 1 and $C_1 = \pm 2/3$ for each possible value of C_1, a different model is obtained.

In order to determine the solution and convenience, $\kappa = c = 1$ will be taken. The EoS of (6) can be rewritten as $P_T = -\Lambda - a(\mu_T - \Lambda)^2$ which is equivalent to $-G_1^1 + \Lambda + a(G_0^0 - \Lambda)^2 = 0$ of which the solution of F is obtained in the form

$$F = \frac{a + C_2 t}{C_2 t + 3C_2 t^3 \Lambda + a + 3t^2 + 3\Lambda t^2 a}, \tag{9}$$

where C_2 is a integration constant. The function σ in (8) can be written in its general form by means of the incomplete elliptic integrals of first and third kind. However, it is a very complex solution; an easier solution can be satisfied by considering that constants $C_2 = 9/(8\sqrt{\Lambda})$ and $a = 1/(8\Lambda)$ where the solution in (9) can be expressed as

$$F = \frac{9t\sqrt{\Lambda} + 1}{(3t\sqrt{\Lambda} + 1)^3}. \tag{10}$$

From the solution (10) and taking into account the Einstein equations, (2) and (3), it is obtained that pressure $P_{BEC} = -8\Lambda/(9t\sqrt{\Lambda} + 1)^2$ and $\mu_{BEC} = 8\Lambda/(9t\sqrt{\Lambda} + 1)$ or else

$$P_T = -\Lambda - \frac{8\Lambda}{(9t\sqrt{\Lambda} + 1)^2} \tag{11}$$

and density μ_T

$$\mu_T = \Lambda + \frac{8\Lambda}{9t\sqrt{\Lambda} + 1}. \tag{12}$$

The function σ in (8) can be written as

$$\sigma = 2\sqrt{\frac{9t\sqrt{\Lambda} + 1}{3t\sqrt{\Lambda} + 1}} - arctanh\left(\frac{6t\sqrt{\Lambda} + 1}{\sqrt{27t^2\Lambda + 12t\sqrt{\Lambda} + 1}}\right) - i\frac{\pi}{2} \tag{13}$$

where $\sigma \in \Re$.

3 Singularities. Kretschmann Invariant

In the study of possible singularities of a given space-time, the Kretschmann invariant is used ($Krets = R_{\alpha\beta\gamma\tau}R^{\alpha\beta\gamma\tau}$). The importance of this invariant has been discussed in [1]. This invariant has the following form for the found solutions

$$\begin{aligned} Krets_{\pm} = {} & \frac{32\left(1 \pm \sqrt{\frac{(3t\sqrt{\Lambda}+1)^3}{9t\sqrt{\Lambda}+1}}\right)}{27t^4} + \frac{8\Lambda(56t\sqrt{\Lambda} + 2)}{t^2(9t\sqrt{\Lambda} + 1)^4} + \\ & + \frac{216\Lambda^2\left(124t^2\Lambda + 72t^3\Lambda^{3/2} + 81t^4\Lambda^2 + 80t\sqrt{\Lambda} + 21\right)}{(9t\sqrt{\Lambda} + 1)^4}, \end{aligned} \tag{14}$$

where the positive sign is used when $C_1 = 2/3$ and the negative, if $C_1 = -2/3$. From the Kretschmann invariant (14), a singularity exists in $t = 0$ when $C_1 = 2/3$ is considered. This singularity is of Kasner type E_{D_1} (see [1]) for which $Krets_+ \to 64/(27t^4) \to \infty$. When $C_1 = -2/3$, Krestchman invariant does not present any singularity of $t = 0$ and tends to $Krets_- \to 324\Lambda^2$. The solution of the metric (1), of (10), (13) and (8) when $t \to 0$ tends to

$$ds^2 \to dt^2 - t^{2/3\pm 2/3}\left(dX^2 + dY^2\right) - t^{2/3\mp 4/3} dZ^2, \tag{15}$$

where $X = e^{\pm 2/3}(3\sqrt{\Lambda}/2)^{\pm 1/3}x$ analogous to y and $Z = e^{\mp 4/3}(3\sqrt{\Lambda}/2)^{\mp 2/3}z$; that is to say, they have an equivalent metric to the Kasner vacuum E_{D_1} or to the flat world E_{D_0} [1]. Hence, for small times, the expansion can be present mostly on the axis "z" when $C_1 = 2/3$ or on the plane "xy" when $C_1 = -2/3$. In both cases, when $t \to \infty$, Kretschmann invariant tends to $Krets_\pm \to 8\lambda^2/3$ which corresponds to the value $Krets$ of a dark energy space-time [1] since the metric solution tends to

$${ds_{isot}}^2 = d\eta^2 - e^{2/3\sqrt{\alpha}\eta}\left(dx'^2 + dy'^2 + dz'^2\right), \tag{16}$$

where $t = e^{\sqrt{3\Lambda}\eta}$ and $x' = ((2\sqrt{3}+3)/(2\sqrt{3}-3))^{\pm 1/6}x$ analogous of y, and $z' = ((2\sqrt{3}+3)/(2\sqrt{3}-3))^{\mp 1/3}z$ which is the solution to a fluid of dark energy in FRWL.

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4 Hubble Parameter and Deceleration

Hubble H and deceleration q parameters for a space-time of Petrov type D have been studied and defined in [7]. Those parameters for the obtained solutions in this work are equal to

$$H = \frac{\left((g_{11}g_{22}g_{33})^{1/6}\right)^{\cdot}}{\sqrt{g_{00}}(g_{11}g_{22}g_{33})^{1/6}} = \frac{(3t\sqrt{\Lambda}+1)^2}{3t\sqrt{(9t\sqrt{\Lambda}+1)(3t\sqrt{\Lambda}+1)}} \tag{17}$$

where metric tensor components have been taking into account $g_{\mu\nu}$ of (1) and

$$q = -(1 + \frac{\dot{H}}{H^2\sqrt{g_{00}}}) = \frac{27t^2\Lambda - 24t\sqrt{\Lambda} - 2}{27t^2\Lambda + 12t\sqrt{\Lambda} + 1}. \tag{18}$$

The Hubble parameter tends to infinity (it is independent of the value of C_1) when $t \to 0$ and when $t \to \infty$, $H \to \sqrt{\Lambda/3}$.

The deceleration parameter q tends to $q \to 2$ when $t \to 0$ regardless of the value of C_1 when $t \longrightarrow \infty$, $q \to -1$ which implies an initially decelerated expansion process that in a time $t = (4+\sqrt{22})/(9\sqrt{\Lambda})$ does not present deceleration either acceleration, so $q = 0$. Then, after that time, it keeps an accelerated phase with an asymptote to $q = 1$.

5 Temperature

Temperature can be determined in the following form [8]

$$\frac{dP_T}{\mu_T + P_T} = \frac{dT}{T}, \tag{19}$$

where T is the temperature of a fluid mixture that for this work is determined exclusively by the BEC.
It is obtained from (19), (12) and (11) that

$$T = \frac{81\Lambda t^2 T_0}{(9t\sqrt{\Lambda}+1)^2}, \tag{20}$$

where T_0 is an integration constant that represents the temperature of which the system tends to as time increases. The system entropy is constant and defined as [8]

$$S = \frac{(\mu_T + P_T)t}{T} = \frac{8\sqrt{\Lambda}}{9T_0}, \tag{21}$$

where μ_T and P_T were obtained from (12) and (11). The entropy $S = S_{BEC} + S_{DE}$ where

$$S_{BEC} = \frac{(\mu_{BEC} + P_{BEC})t}{T} = \frac{8\sqrt{\Lambda}}{9T_0} \; and \; S_{DE} = \frac{(\mu_{DE} + P_{DE})t}{T} = 0, \tag{22}$$

so the entropy is null and different from zero for the BEC due to dark energy. From (20), it is noted that when $t \to 0$, temperature tends to zero, so a BEC should have an essential presence in the starts of the universe. The foregoing also is determined from (12), and as time increases, density $\mu_T \to \Lambda$, but for $t \to 0$, $\mu_T \to 9\Lambda$. In other words, almost a 89% of the total volumetric density of the system is initially due to the BEC (according to the chosen constants in this work).

6 Conclusions

In this work, solutions of a mixture of dark energy and of an attractive BEC were obtained. They were different from each other because of the way in which the universe intensively expands in its early times either by an axis or by a plane. Those differences determined whether the model universe (the solution) is singular or not; it is singular when it expands more on an axis and it is not when it expands more on a plane even though they do not keep significant differences to high times. In the solutions, the volumetric energy density μ_T and pressure P_T do not differ from each other and for them, both magnitudes are finite in all $t \geq 0$. For the type solution studied, BEC density is the most important one in the system in its start since it represents almost a 89% of the total density value μ_T, but it losses its importance as time increases in a way that (according to O(1/t) order), $\mu_{BEC} \to 8\sqrt{\Lambda}/(9t)$ and $P_{BEC} \to 0$ which represents that a BEC behaves as a fluid of dust type [1] for higher values of t. Additionally, temperature is null at the start what explains the density value of a high BEC for those times. Solutions tend to be equivalent to the FRWL universe for dark energy changing asymptotically from an anisotropic and homogeneous regime of a fluid mixture of a BEC and dark energy to an isotropic and homogeneous one. After studying the Hubble and deceleration parameters, it was established that when $t \to 0$, the Hubble parameter gets indefinite in the start, but as time increases, it tends to a constant value equal to the one from dark energy; meanwhile, for the deceleration parameter, the universe initially decelerates and as time increases, it continuously changes to an accelerated process of expansion.

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