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Scalar Field Equation in a Class of LTB Cosmologies: Normal Modes, Field Quantization and Instantaneous Particle Creation

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Abstract

The scalar field equation is studied in Lemâitre-Tolman-Bondi cosmological model. The equation is solved by variable separation. This is possible within a suitable class of LTB cosmological models characterized by a single arbitrary integration cosmological function. The separated angular equation integrates exactly. The separated time equation is such that the associated Wronskian equation has a simple form. The separated radial equation strictly depends on the choice of the arbitrary cosmological integration function. It can be recast into the form of eigenvalue problem of a Weyl Stone operator that results to be essentially self adjoint in a suitable non void class of LTB cosmological models. In turns this finally ensures the existence of complete set of normal modes that are the basis for a canonical quantization of the scalar field. A time contiguous set of normal modes is then constructed. By the calculation of the corresponding Bogolubov coefficients the instantaneous rate of particle creation is then obtained in complete agreement with analogous results.

Keywords: LTB cosmology - Scalar field equation - Separation - Normal modes - Quantization - Particle creation

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1 Introduction

Particle creation in curved space time was originally considered and studied by L. Parker in [7, 8], and in references therein, in the context of flat Robertson Walker (RW) space time. The result, that was obtained by the canonical quantization scheme for general spin field, has been both widely developed and extended to general space time. The procedure is now treated in books and reviews devoted to field quantizatin in curved space time (e.g. [1, 4, 9, 2]; for a review of the particle creation problem see also [3]).

Recently it has been noted that the particle production rate can be instantaneously calculated in RW space time. It results proportional to $\dot{R}/R$, $R(t)$ being the radius of the Universe. The result was given for spin $0, 1/2, 1$ field [13, 15, 14]. Similar studies have been performed for spin $0, 1$ fields in the context of Lemâitre Tolman Bondi (LTB) space time [16]. Generalized expression of the particle production rate has also been used to propose extension of both the Standard Cosmology and of the LTB cosmological model to include particle creation [19, 17]. In the mentioned studies the quantization procedure is based on the knowledge of a complete set of normal mode solutions of the field equation. Such normal modes were not always explicitly given in previous studies nor their existence proved in general. In case of the scalar field, there form is explicitly known in the RW space time (results have been collected in [10]), while in LTB space time and LTB cosmology the completeness of the normal modes has been proved only for special cases of the cosmological integration function, while are only supposed to exist in a general LTB cosmology (see e. g., [12]).

The object of the present paper is to fill that lack in case of scalar field. To that end it is useful to recall that the equation defining the LTB cosmological model can be solved exactly. It depends on two arbitrary integration function of the radial coordinate.

Under a suitable relation between the two cosmological integration functions, the scalar field equation results to be separable (as well as those of arbitrary spin field [11, 12, 23]). The separated angular equation is exactly integrable. The separated time equation is such that its Wronskian equation easily integrates. Instead the separated radial equation still depends on the arbitrary cosmological integration function. The separated radial equation can be put into the form of a differential operator eigenvalue problem that, by the general Weyl Stone theorem, results , under suitable assumptoins, to be essentially selfadjoint . The corresponding complete set of eigenvector, finally ensures the existence of a complete set of normal modes that is the basis for the canonical quantization of the scalar field.

The form of the normal modes allows the definition of a set of "time contiguous" normal modes. By those states, the Bogolubov coefficients are calculated.

In turn they allow the calculation of the expression of the instantaneous rate of particle creation. This is obtained by evaluating the expectation of the particle operator number. The result is coherent with the previously mentioned ones.

2 Assumptions and preliminary results

The scalar field equation of a (minimally coupled) complex scalar field ϕ in a general curved space time of metric tensor $g_{\alpha\beta}$ has the form

$$\nabla_\alpha \nabla^\alpha \phi(x) + \mu^2 \phi(x) = 0 \tag{1}$$

μ being the mass of the field particle. Its quantization is based on the knowledge of a complete set of solution of the equation (1) orthonormal in the scalar product defined by [1]:

$$(\phi_1, \phi_2) = \int J_\nu(\phi_1, \phi_2)\, n^\nu |g|^{1/2}\, d\Sigma \tag{2}$$

$$= \int_{t_0} J_t(\phi_1, \phi_2) |g|^{1/2} d_3 x \tag{3}$$

where $J_\nu = -i(\phi_1 \nabla_\nu \phi_2^* - \phi_2^* \nabla_\nu \phi_1)$ is a conserved four vector (the four current if $\phi_1 = \phi_2$), Σ a space like hyper surface and n^ν a future directed unit vector orthogonal to Σ and g the determinant of $g_{\alpha\beta}$. The expression (3) gives an easier way to select a complete set of ortho-normal solutions of equation (1) satisfying

$$(\phi_\alpha, \phi_\beta) = \delta_{\alpha\beta}, \quad (\phi_\alpha^*, \phi_\beta^*) = -\delta_{\alpha\beta}, \quad (\phi_\alpha, \phi_\beta^*) = 0 \tag{4}$$

in order to proceed to the co-variant quantization in analogy to the Minkowski space time case.

The object is now of first providing a set of normal modes of the scalar field in the context of a suitable class of LTB cosmological models.

As well known the LTB cosmology corresponds to a Universe filled of with freely falling dust like matter of metric tensor (e.g. [6]):

$$g_{\mu\nu} = diag\Big\{1;\, -e^{\Gamma(t,r)};\, -Y^2(t,r);\, -Y^2(t,r)(\sin\theta)^2\Big\} \tag{5}$$

The corresponding Einstein field equation can be partially integrated so that one is left with the equations

$$e^\Gamma = \frac{Y'^2}{1+2E(r)}, \qquad \frac{\dot{Y}^2}{2} - \frac{M(r)}{Y} = E(r) \tag{6}$$

$$M(r) = 4\pi G \int_0^r dr\, Y^2\, Y' \rho(t,r) \tag{7}$$

$\rho(t,r)$ is the energy density of the dust matter and $M(r), E(r)$ arbitrary integration functions. The Kepler like equation can be exactly integrated in parametric form (e.g. [6]). For the following consideration it is now chosen:

$$(|2E|)^{\frac{3}{2}} = M, \quad E \neq 0 \quad (G = 1) \tag{8}$$

By such condition the cosmological model depends on one only arbitrary function, say E. The advantage is that it ensures the separability not only of the scalar field equation but also the separability of the arbitrary spin field equation in LTB cosmology (see e. g., [11, 12, 16, 18, 23]). Under condition (8) the solution of the Kepler like equation reduces, by also assuming the initial time $t_0(r) = 0$, to:

$$t = f(\eta), \quad Y = \sqrt{2|E|}\ \xi(\eta), \quad \xi(\eta) = f'(\eta) \tag{9}$$

$$f(\eta) = \sinh \eta - \eta, \quad \eta > 0, \quad E > 0 \tag{10}$$

$$f(\eta) = \eta - \sin \eta, \quad 0 < \eta \leq 2\pi, \quad E < 0 \tag{11}$$

In turn this implies that the density of the cosmological model results to be spatially uniform:

$$\rho(t,r) = \frac{M'(r)}{Y^2(t,r)Y'(t,r)} \equiv \frac{3}{\xi^3)\eta)} \tag{12}$$

This loss of generality on the cosmological model, is however balanced from the fact that condition (8) is a sufficient condition to finally prove the existence of a complete set of normal modes.

Based on the fact that

$$\tau = (2|E|)^{1/2} \int \frac{1}{Y(r,t)} dt \equiv \eta, \tag{13}$$

that follows from (9), in the following it will be useful to pass from the coordinates (r,t) to the coordinates (r,τ)

3 Normal modes

On account of the spherical symmetry of the space-time metric the angular dependence of the scalar field solution can be separated and integrated in the general metric (5). By setting $\phi = \chi(\theta,\varphi)\psi(r,t)$, from (1), (5) one gets

$$\chi = Y_{lm}(\theta,\varphi), \quad l = 0,1,2,...,m = -l,-l+1,...,0,1,..,l \tag{14}$$

$$\psi_{tt} - \frac{\psi_{rr}}{e^{\Gamma}} + \frac{1}{e^{\Gamma}}\Big(\frac{\Gamma'}{2} - 2\frac{Y'}{Y}\Big)\psi_r + \Big(\frac{\dot{\Gamma}}{2} + 2\frac{Y'}{\dot{Y}}\Big)\psi_t + \Big(\frac{\lambda}{Y^2} + \mu^2\Big)\psi = 0 \tag{15}$$

where $\lambda = l(l+1)$ corresponds to the angular separation constant that plays the role of eigen value of the eigenvalue equation to which the separated angular equation can be reported; Y_{lm} the usual spherical harmonics [12] and $\dot\Gamma = \partial_t\Gamma, \Gamma' = \partial_r\Gamma$.

Further separation follows by restricting to the LTB cosmology satisfying condition (8). By first passing to the variables r, τ and then by setting $\psi(r,\tau) = T(\tau)R)(r)$ one obtains from (15) ([12]):

$$\ddot T + \frac{2\sinh\tau}{\cosh\tau - 1}\dot T + [\mu^2(\cosh\tau-1)^2+\sigma]T = 0 \tag{16}$$

$$R'' - \Big[\frac{E''}{E'} - \frac{E'(3+8E)}{2E(1+2E)}\Big]R' - \frac{E'^2}{2E(1+2E)}\Big(\frac{\lambda^2}{2E} - \sigma\Big)R = 0 \tag{17}$$

σ the separating constant ($\dot T = dT/d\tau$)

We denote now $\phi_\alpha = T_\sigma(\tau)\,Y_{lm}(\theta,\varphi)\,R_{l\sigma}(r),\ \alpha \equiv (l,m,\sigma)$, the solution of the scalar field in the LTB model under condition (8). The product of these solution according to (3) gives:

$$(\phi_\alpha, \phi_{\alpha'}) = \int J_t(\phi_1,\phi_2)|g|^{1/2}d_3x \tag{18}$$

$$= \delta_{ll'}\delta_{mm'}(T_\sigma\dot T^*_{\sigma'} - T^*_{\sigma'}\dot T_\sigma)\frac{\xi(\eta)^3}{i}\int_I dr\frac{\big[(2|E|)^{\frac12}\big]'\,2|E|}{\sqrt{1+2E}}R_{l\sigma}R_{l\sigma'} \tag{19}$$

$$= \delta_{ll'}\delta_{mm'}\delta_{\sigma\sigma'} \tag{20}$$

because, by a suitable choice of the integration constant of the wronskian equation of (16), and then by passing to the variable t, one has then (here $\dot T = dT/dt$):

$$T_\sigma(t)\dot T^*_\sigma(t) - T^*_\sigma(t)\dot T_\sigma(t) = i[\xi(\eta)]^{-3} \tag{21}$$

and by assuming, as it will be presently seen to be possible,

$$\int_I dr\frac{\big[(2|E|)^{1/2}\big]'\,2|E|}{\sqrt{1+2E}}R_{l\sigma}R_{l\sigma'} = \delta_{\sigma\sigma'} \tag{22}$$

It has been denoted $I = (0,\infty)$ if $E > 0$ or $I = (0, r_0)$ determined by the condition $1 + 2E \geq 0$; $\delta_{\sigma\sigma'}$ is the Kronecker (possibly the Dirac) delta.

The eq. (17) can be recast into the eigen value-problem:

$$\hat A R \equiv \frac{1}{k(r)}\Big[-(p(r)R'(r))' + q(r)R(r)\Big] = \sigma R(r) \tag{23}$$

$$k = \frac{E^{\frac12}E'}{2(1+2E)^{\frac12}}, \quad p = \frac{(E^3(1+2E))^{\frac12}}{E'}, \quad q = \frac{\lambda E'}{(E(1+2E))^{\frac12}} \tag{24}$$

According to a suitable choice of $E > 0$ and hence of $k(r), p(r), q(r)$, the Weyl-Stone operator $\hat A$ results to be essential selfadjoint in the Weyl subspace U of $L^2((0,\infty); k(r)dr)$ ([5] Ch. 14, Th. 1). The set of the positive E functions for which this is indeed possible is not void, as shown in [12]). The condition (22) is then realized by the complete set of the eigen vector of $\hat{A}$.

In turn the corresponding set of solution $\{\phi_\alpha = T_\sigma(\tau)\, Y_{lm}(\theta,\varphi)\, R_{l\sigma}(r)\}$ furnishes a complete set of normal modes of the scalar field equation (1) satisfying the product conditions (4).

4 Scalar field quantization

The scalar field quantization can be implemented by expanding the wave scalar field operator $\phi(x)$ in terms of creation and annihilation operators a_i^+, a_j ($[a_i, a_k^+] = \delta_{ik}$, $[a_i^+, a_k^+] = [a_i, a_k] = 0$) that is:

$$\phi(x) = \sum_j [a_j \phi_j(x) + a_j^+ \phi_j^\star(x)] \tag{25}$$

(here e., g. $\sum_j = \sum_{lm} \int d\sigma$) where $\{\phi_j(x)\}$ is a set of normal modes solutions of the scalar field equation satisfying condition (4). The corresponding Fock space F is then constructed from the assumption that there is a state $|0>$ (the empty state) such that $a_j|0>= 0\ \forall j$, ($a_k^+|0>\neq 0$ for some k) and repeatedly applying creation particle mode operators (e., g. [2, 4, 1, 9]).

Suppose $\hat{\phi}_\alpha(x)$ is a second set of normal modes with associated $|\hat{0}>$ empty state (and Fock space $\hat{F}$):

$$(\hat{\phi}_\alpha, \hat{\phi}_{\alpha'}) = \delta_{\alpha\alpha'} \qquad (\hat{\phi}^\star_\alpha, \hat{\phi}^\star_{\alpha'}) = -\delta_{\alpha\alpha'} \qquad (\hat{\phi}_\alpha, \hat{\phi}^\star_{\alpha'}) = 0 \tag{26}$$

Being both $\{\hat{\phi}_\alpha\}$ and $\{\phi_\beta\}$ complete sets, one has

$$\phi(x) = \sum_j [\hat{a}_j \hat{\phi}_j + \hat{a}^+ \hat{\phi}^\star] \tag{27}$$

$$\hat{\phi}_j = \sum_i [\alpha_{ji}\phi_i + \beta_{ji}\phi_i^\star], \quad \alpha_{ji} = (\hat{\phi}_j, \phi_i), \tag{28}$$

$$\phi_h = \sum_l [-\alpha^\star_{lh}\hat{\phi}_l - \beta_{lh}\hat{\phi}^\star_l], \quad \beta_{ji} = -(\hat{\phi}_j, \phi^*) \tag{29}$$

$$a_j = \sum_l [\pm\alpha_{lj}\hat{a}_l \mp \beta^\star_{lj}\hat{a}^+_i], \quad \hat{a}_j = \sum_i [\alpha^\star_{lj} a_i + \beta^\star_{ji} a_i^+] \tag{30}$$

α_{ji}, β_{ji} the Bogolubov coefficients (see e., g., [1]). By setting $A = \{\alpha_{ji}\}$, $B = \{\beta_{ji}\}$, the Bogolubov coefficients satisfy the constraint $AA^+ - BB^+ = 1$, $BA^T - AB^T = 0$.

5 Particle creation by Universe expansion

In analogy to what suggested in [14], it is possible to define n the LTB cosmological model of Section 3 a new set of normal modes for the scalar field, contiguous to those previously determined . By the normal modes $\{\phi_i(x)\}$of Section 3 one can define a new set of normal modes $\{\hat{\phi}_i\}$. Indeed by setting

$$\hat{\phi}(t) = \frac{Y(t+\tau, r, \theta, \varphi)}{Y(t, r, \theta, \varphi))}\phi^*(t+\tau, r, \theta\varphi) = \frac{\xi(t+\tau)}{\xi(t)}\phi^*(t+\tau, r, \theta\varphi), \tag{31}$$

one has in the product (18) and by using (21),

$$(\hat{\phi}_i, \hat{\phi}_k) = \delta_{ik}(-i)\Big[T_i^*(t+\tau)\dot{T}(t+\tau) - T_i(t+\tau)\dot{T}^*(t+\tau)]\xi^3(t+\tau)\Big] = \delta_{ik} \tag{32}$$

Similarly one can verify all the other relations (26).

It is possible to apply the quantization scheme of the previous Section to calculate the Bogolubov coefficients relative the normal modes $\{\phi_i(x)\}$ and $\{\hat{\phi}_i(x)\}$ determined by the cosmological assumpton (8).

An observable of interest is the number of particle mode operator $N_h = a_h^+ a_h$. One has from (30)

$$< \hat{0}|N_h|\hat{0} >= ||a_h|\hat{0} > ||_{\hat{F}}^2 = \Big|\Big| \sum_l \beta_{lh}^\star \hat{a}_l^+|\hat{0} > \Big|\Big|_{\hat{F}}^2 = \sum_l |\beta_{lh}^*|^2 \tag{33}$$

$$< 0|\hat{N}_h|0 >= \Big|\Big| \sum_j \beta_{hj}^\star a_j^+|0 > \Big|\Big|_F^2 = \sum_j |\beta_{hj}|^2 \tag{34}$$

while $< 0|N_h|0 >=< \hat{0}|\hat{N}_h|\hat{0} >= 0$. Therefore there are particle in the h-mode "present" in $|\hat{0} >$ that are not present in $|0 >$ and particle in the h-mode "present" in $|0 >$ that are not present in $|\hat{0} >$.

This means that here are h-mode particles that "annihilate" at time t and there are h-mode particles that are "created" at time $t+\tau$ because the empty states $|0 >, |\hat{0} >$ correspond to those times. In terms of Bogolubov coefficients

$$\beta_{ik} = -i\Big[\xi(t+\tau)\xi(t)\Big]^{3/2}[T_k^*(t+\tau)\dot{T}(t)_k - \dot{T}_k^*(t+\tau)T(t)_k]\delta ik \tag{35}$$

by using (33) the expectation of created and annihilated particles N^+, N^- calculated respectively with $g = g(t+\tau), g = g(t)$, can be expressed by

$$N_k^- = \Big[\xi(t+\tau)\xi(t)\Big]^3\Big|[T_k^*(t+\tau)\dot{T}_k(t) - \dot{T}_k^*(t+\tau)T_k(t)]\Big|^2, \tag{36}$$

$$N_k^+ = \Big[\frac{\xi(t+\tau)}{\xi(t)}\Big]^3\xi^6(t+\tau)\Big|\Big[T_k^*(t)\dot{T}_k](t) - \dot{T}_k^*(t)T_k(t)\Big]\Big|^2 \tag{37}$$

The expectation of the net production of particles per unit time is then, by using also eq. (21):

$$n_k(t) = \lim_{\tau\to 0}\frac{N_k^+(\tau) - N_k^-(t)}{\tau} = 6\frac{\dot{\xi}(t)}{\xi(t)} = 6\frac{\xi'(\eta)}{\xi^2(\eta)} \tag{38}$$

Therefore, under assumptipn (8), we obtain explicitly $n_k(t) \propto \dot{Y}/Y$. Such expression was assumed, by abstraction, to furnish also the particle creation rate in cosmological model for which Y is not necessarily a factorized function in the r, t dependence [22].

6 Remarks and comments

The content of the present paper represents both an improvement of previous ideas and developments contained in [12, 14]. With respect to the paper [12], a characterization is here given for the class of the positive cosmological E functions for which there exist a complete set of normal modes of the scalar field besides the single case there solved. Moreover, the present result represents a complete discussion of the quantization scheme in LTB cosmology previously performed in an explicit form in the Robertson Walker space time [14].
There are some comments:

i) From a mathematical point of view, a further improvement would be the determination of the solutions (and the corresponding normal modes) relative to the equation (15) without assumption (8). It is expected that this would give qualitative new aspects.

ii) For what concerns the number of created particle of arbitrary mode in a volume dV, it results again proportional to $\dot{Y}/Y$ because the expression (38) does not depend on the mode nor on r. Hence the density of $\rho_c(t)$ of created particle at time t is of the form $\rho_c(t) = \alpha\dot{Y}/Y$ where α a cosmological parameter to be possbly determined. In view of a formulation of a standard like cosmological model, the density of the created particles should be added to the existing density $\rho(t)$ of dust matter. (Such point of view was adopted in RW space time by formulating an extended standard cosmological model to include particle creation [19, 20, 21]). In case of the LTB model, a possible more general point of view could be of assuming the dust matter to have also non zero pressure and to obey a generalization of the result (38) [16, 17].

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