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Models of H-atom in Lemaitre Tolman Bondi Cosmology from Dirac Equation with 'Curved' Coulomb Potential

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Abstract

The Dirac equation with Coulomb potential is studied in Lemaître Tolman Bondi cosmological model. Angular separated equations are obtained, by using the Newman-Penrose formalism, that admit of known solution previously determined. One is left with a pair of coupled first order partial differential equations in the radial and time coordinates containing the "curved" Coulomb potential. The equations are further reduced, under a special assumption on the spatial configuration of the cosmological model and by considering the cosmological background essentially fixed under atomic time intervals. The equations are exactly solved, in a parameter dependent cosmological LTB model, under two approximated expressions for the "curved" Coulomb potential: a first one that is essentially the conventional Coulomb potential and a second one that is its first order approximation form. In both cases, by a Schrôdinger-like quantization requirement, the energy spectrum of the H-atom is obtained exactly. The results are similar to those previously given for the Robertson Walker space time case.

Keywords: LTB cosmology, Curved Coulomb potential, Dirac equation, Separation, H atom, Discrete energy spectrum

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1 Introduction

The observation of the spectral lines of the Hydrogen atom and their modification by gravity are of interest in astrophysics and cosmology. The H-atom can indeed be used to localize Hydrogen distribution in the universe. It could be used also as a probe to detect gravitational fields. The theoretical previsions for that observation are generally based on the study of Dirac equation with potential in curved space time. This involves the knowledge of the explicit expression of the Coulomb potential in the considered space time. Such expression can be obtained perturbatively in a general space time [12, 13, 14, 3, 9] or explicitly determined by solving the Maxwell equation for the electromagnetic field of a fixed charged point (e. g., [11, 7, 20]) (From the point if view of Path integral methods one can see, e. g., [1] and References therein). By the above considerations it seems of interest to treat the problem directly in an explicit cosmological model.

In the present paper the study is performed in Lemâitre Tolman Bondi cosmological model. That model can be exactly integrated, the solution depending on two arbitrary integration functions. This allows to describe a very wide class of gravitational situations. Moreover in such cosmologies, the expression of the "curved" Coulomb potential, again depending on the two arbitrary cosmological integration function, may have an explicit expression very far from the conventional one (See, e. g., [20]).

The H-atom is here described by the Dirac equation with the "curved" Coulomb potential. By the Newman Penrose formalism [15] based on a previously considered null tetrad frame, the angular dependence of the solution of the Dirac equation with general central potential is first factored out and integrated [19]. One is then left with a pair of coupled first order partial differential equations in the t, r variables.

The form of the equations is further reduced by assuming the space time to be static, a reasonable condition during the time intervals the atomic interactions take place, far from the initial time of the cosmological background evolution. Further simplification of the equations is then obtained by a change of variables. One is finally left with two coupled first order ODE's that generalize the analog ones of the Robertson Walker space time [21].

The final equation are exactly solved, in a special parameter dependent cosmological background, by the method of solution that was employed for similar equations in Minkowski space time [5]. The energy spectrum of the H-atom is then explicitly determined by a Schrôdinger like quantization for both the basic form of the "curved" Coulomb potential as well as for its first order approximation. In both cases the calculations are performed exactly and not perturbatively. The spectra obtained differ from those of the Robertson Walker space time [21] only by the dependence of the electric interacting constant on the assumed cosmological parameter.

2 Definitions and assumptions

The object being of studying the Hydrogen atom in Lemâitre-Tolman-Bondi (LTB) cosmology, the Dirac equation with central potential is first formulated in the LTB space time. Accordingly the Dirac field is described by a pair of spinor field $(\phi_A, \chi^{B'})$ (e. g., [6]). The study, that is developed by the Newmnan Penrose formalism, simplifies if the Dirac field is described by the pair of spinors fields (P_A, Q^B) with $P_A \equiv \phi_A$, $\bar{Q}^{B'} \equiv -\chi^{B'}$ ([4]). Accordingly, the Dirac equation for a particle of mass μ and charge $-e_0$ subjected to the potential $V_{AX'}$, reads :

$$(\nabla_{AX'} + iV_{AX'})P^A + i\mu_\star \bar{Q}_{X'} = 0 \tag{1}$$
$$(\nabla_{AX'} - i\bar{V}_{AX'})Q^A + i\mu_\star \bar{P}_{X'} = 0 \quad (\mu_\star = \mu/\sqrt{2}) \tag{2}$$

that will be studied in the space time of metric tensor $g_{\mu\nu}$ given by [16]:

$$g_{\mu\nu} = diag\Big\{1;\ -e^{\Gamma(t,r)};\ -Y^2(t,r);\ -Y^2(t,r)(\sin\theta)^2\Big\} \tag{3}$$

The equations (1), (2) can be expanded in terms of the directional derivatives and spin coefficients . If the null tetrad frame defined in [17] is adopted, they can be separated under the conditions $V_{AX'} = V_{AX'}(t,r)$, $V_{01'} = V_{10'}$, by setting

$$P^A \equiv \frac{1}{Y}\Big(H_1(t,r)S_1(\theta),\ H_2(t,r)S_2(\theta)\Big)e^{im\varphi} \tag{4}$$
$$\bar{Q}^A \equiv \frac{1}{Y}\Big(-H_1(t,r)S_2(\theta),\ H_2(t,r)S_1(\theta)\Big)e^{im\varphi}, \quad m = 0, 1, 2, ... \tag{5}$$

This leads to a separate angular eigenvalue problem that can be exactly integrated. There results that S_1, S_2 are essentially given by Jacobi polynomials $S_{ilm}(\theta)$, $i = 1, 2$; $l = 0, 1, 2, ..$ ([10, 18, 19, 21]).
One is then left with the equations in the D, Δ directional derivatives, spin coefficients ϵ and t, r variables:

$$\sqrt{2}\,(D + \epsilon + iV_{00})H_1 = \Big(i\mu + \lambda/Y\Big)H_2 \tag{6}$$
$$\sqrt{2}\,(\Delta + \epsilon + iV_{11})H_2 = \Big(i\mu - \lambda/Y\Big)H_1 \tag{7}$$
$$\sqrt{2}\,D = \partial_t + e^{-\Gamma/2}\,\partial_r \tag{8}$$
$$\sqrt{2}\,\Delta = \partial_t - e^{-\Gamma/2}\,\partial_r \tag{9}$$
$$\epsilon = \dot{\Gamma}/2^{5/2} \tag{10}$$

where $\lambda^2 = l(l+1)$, $l = 0, 1, 2, ..$ is the eigenvalue of the separated angular equations [19]. In the following the spinor $V_{AX'}$ will be assumed to be of the form [20]:

$$V_{AA'} = \sigma^{\alpha}_{AA'} V_\alpha = \frac{V}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Big(V_\alpha \equiv (V,0,0,0)\Big) \tag{11}$$

$$V_0 = V(t,r) = e_0^2 \int \frac{e^{-\Gamma/2}}{Y^2} dr \tag{12}$$

σ_0 the t σ-matrix of the assumed null tetrad frame. V is the "curved" Coulomb potential energy of the given point of mass μ and charge $-e_0$ interacting with a point of charged e_0 in the origin of the coordinates (e. g., [20]).

It is useful to note that the "curved" Coulomb potential always corresponds to an attractive force. Indeed $-\partial V_0/\partial r = -e_0^2 e^{\Gamma/2}/Y^2 < 0$ on account of the physical interpretation of Γ, Y in terms of real functions.

3 The LTB cosmological model

The equations (6), (7) are now studied for a specific gravitational model, namely in Lemâitre-Tolman-Bondi cosmological model. As it is well known (e. g., [8]) the LTB cosmology corresponds to a spherically symmetric space time filled freely falling dust matter of zero pressure in the coordinate system (3). The model can be exacly integrated. The cosmological solution depends on two arbitrary integration function $E(r), M(r)$ that represent the energy and the mass of a one dimensional gravitational equation to which the model can be reduced. Here we are interested in the case $E \neq 0$. The cosmological model can be equivalently described by the equations

$$e^\Gamma = \frac{Y'^2}{1+2E(r)}, \quad \frac{\dot{Y}^2}{2} - \frac{M(r)}{Y} = E(r) \tag{13}$$

$$M(r) = 4\pi G \int_0^r dr\, Y^2\, Y' d(t,r) \tag{14}$$

where $d(t,r)$ is the energy density of the dust matter. For the present purposes it is convenient to give the solutions of the model in parametric form [8]:

$$t = x(r)\xi^t(\eta), \qquad Y = y(r)\xi(\eta) \tag{15}$$

$$x(r) = G\frac{M(r)}{(2|E|)^{3/2}}, \quad y(r) = \frac{M(r)}{2|E|} \qquad (E \neq 0) \tag{16}$$

$$\xi = \cosh\eta - 1, \quad \xi^t = \sinh\eta - \eta, \quad \eta \geq 0 \quad (E > 0) \tag{17}$$

$$\xi = 1 - \cos\eta, \quad \xi^t = \eta - \sin\eta, \; 0 \leq \eta \leq 2\pi \quad (E < 0) \tag{18}$$

To further study the equations (6), (7) we now pass from the coordinate (t,r) to the coordinates (τ, s) defined by

$$\tau = \int \frac{2|E|)^{1/2}}{Y} dt \equiv \eta \tag{19}$$

$$s(r) = \int \sqrt{\frac{2|E|}{1+2E}}\,\frac{Y}{Y'}dr \tag{20}$$

In the new coordinates the equations (6), (7) read then ($\dot{\xi} = d\xi/d\eta, G = 1$):

$$(\partial_\eta + \partial_s)H_1 + \Big(\frac{1}{2}\frac{\dot{\xi}}{\xi} + i\sqrt{2}V\frac{M\xi}{(2|E|)^{\frac{3}{2}}}\Big)H_1 = \Big(\frac{i\mu M\xi}{(2|E|)^{\frac{3}{2}}} - \frac{\lambda}{(2|E|^{\frac{1}{2}})}\Big)H_2 \tag{21}$$

$$(\partial_\eta - \partial_s)H_1 + \Big(\frac{1}{2}\frac{\dot{\xi}}{\xi} + i\sqrt{2}V\frac{M\xi}{(2|E|)^{\frac{3}{2}}}\Big)H_1 = \Big(\frac{i\mu M\xi}{(2|E|)^{\frac{3}{2}}} + \frac{\lambda}{(2|E|)^{\frac{1}{2}}}\Big)H_2 \tag{22}$$

By further setting

$$H_1 - H_2 = r\,f(\eta, s), \qquad H_1 + H_2 = r\,g(\eta, s) \tag{23}$$

the last equations become ($f_s = \partial f/\partial_s$):

$$g_\eta + f_s + \Big(\frac{r_s}{r} - \frac{\lambda}{(2|E|)^{\frac{1}{2}}}\Big)f + g\Big(\frac{1}{2}\frac{\dot{\xi}}{\xi} + \frac{i\sqrt{2}VM\xi}{(2|E|)^{\frac{3}{2}}}\Big) - \frac{i\mu M\xi}{(2|E|)^{\frac{3}{2}}}\Big) = 0 \tag{24}$$

$$g_s + f_\eta + \Big(\frac{r_s}{r} + \frac{\lambda}{(2|E|)^{\frac{1}{2}}}\Big)f + g\Big(\frac{1}{2}\frac{\dot{\xi}}{\xi} + \frac{i\sqrt{2}VM\xi}{(2|E|)^{\frac{3}{2}}}\Big) + \frac{i\mu M\xi}{(2|E|)^{\frac{3}{2}}}\Big) = 0 \tag{25}$$

This is the most general form to which the Dirac equation with spherically symmetric potential can be reduced in the LTB cosmology. Further progress can be obtained only by assigning the cosmological integration functions E, M.

4 Approximated H-models

The Dirac equation describing microscopical systems and having here as an object to study the Hydrogen atom, it seems reasonable to assume as a basic approximation, the cosmological background to be fixed under time intervals the atomic interactions take place. Here it is assumed

$$\frac{\dot{\xi}(\eta)}{\xi(\eta)} \approx 0 \tag{26}$$

for $\eta = \eta_0$ sufficiently large so to have $\dot{\xi}(\eta_0) \approx 0$.

Accordingly, the time dependence can be factored out by setting $f(\eta, s) = \exp(ik\xi_0\eta)f(s)$, $g(\eta, s) = \exp(ik\xi_0\eta)g(s)$. The equations (24), (25) become then:

$$f' + \Big(\frac{r_s}{r} - \frac{\lambda}{(2|E|)^{\frac{1}{2}}}\Big)f + ig\Big[k\xi_0 - \Big(\mu\xi_0 - \sqrt{2}V\xi_0\Big)\frac{M}{(2|E|)^{\frac{3}{2}}}\Big] = 0 \tag{27}$$

$$g' + \Big(\frac{r_s}{r} - \frac{\lambda}{(2|E|)^{\frac{1}{2}}}\Big)g + if\Big[k\xi_0 + \Big(\mu\xi_0 - \sqrt{2}V\xi_0\Big)\frac{M}{(2|E|)^{\frac{3}{2}}}\Big] = 0 \tag{28}$$

Under the present assumptions r_s follows by reversing the relation (20). The equations are the analog of eqs. (19) in [19] and of (14), (15) in [21].

The equations (27), (28) are now studied for different choice of the integration functions E, M. This involves unconventional expression of the Coulomb field. To simplify the form of the equations one can directly chose

$$(|2E|)^{\frac{3}{2}} = M \qquad (G = 1) \tag{29}$$

Under such condition the density $d(t, r)$ of the dust matter does not depend on r (it is therefore constant if (26) holds true). Indeed, by (29), $t = \xi^t(\eta)$ or $\eta = (\xi^t)^{-1}(t)$ and $Y = y(r)\xi(\eta) \equiv (2|E|)^{1/2}\xi(t)$ so that

$$d = \frac{M'}{4\pi Y^2 Y'} = \frac{1}{4\pi\xi^3(t)} \tag{30}$$

As an application of the scheme under conditions (26), (29) suppose $2E(r) = (ar)^2$. Then:

$$y(r) = (2|E|)^{1/2} = ar, \quad a > 0 \tag{31}$$

$$M(r) = (ar)^3 \tag{32}$$

$$s(r) = \chi \int \frac{d(ar)}{\sqrt{1 + a^2 r^2}} = \chi \sinh^{-1} ar \tag{33}$$

$$r = \frac{1}{a}\sinh s \cong \frac{1}{a}s, \quad s^2 << 1 \tag{34}$$

$$V = \frac{\chi}{a^2}\Big[-\frac{\sqrt{1 + a^2 r^2}}{ar} + \log(ar + \sqrt{1 + a^2 r^2})\Big] \tag{35}$$

$$\cong \frac{\chi}{a^2}\Big[-\frac{1}{s} + s\Big], \quad \chi = \frac{e_0^2}{\xi_0^3} \tag{36}$$

$$\cong -\frac{\chi}{a^2}\frac{1}{s}, \quad r^2 \approx 0,\ s^2 \approx 0 \tag{37}$$

Under these assumptions and by using the approximation (37) for the Coulomb potential energy, the equations (27), (28) read then

$$f'(s) + \frac{1 - \lambda}{s}f(s) + ig(s)\Big(k_0 - \mu_0 + \frac{V\xi_0}{a^2}\Big) = 0 \tag{38}$$

$$g'(s) + \frac{1 + \lambda}{s}g(s) + if(s)\Big(k_0 + \mu_0 + \frac{V\xi_0}{a^2}\Big) = 0 \tag{39}$$

where $k_0 = k\xi_0, \mu_0 = \mu\xi_0$.

With the choice (37) of V these equations are of the same form, in the variable s, of those in the variable r discussed in [19], relatively to the Robertson Walker metric, after the substitution $\chi \longrightarrow \chi/a^2$. A Schrodinger like interpretation of the solutions [2, 21] allows then again to determine a discrete energy spectrum of the H-atom of the form

$$\frac{k}{\mu} = \left[1 + \frac{(\chi/a^2)^2}{(n + \sqrt{\lambda^2 - (\chi/a^2)^2})^2}\right]^{-\frac{1}{2}}, \quad n = 1, 2, ... \tag{40}$$

By increasing a the energy levels increasingly shift towards the limit value 1.

If instead one considers the eqs. (38), (39) with the expression (36) of V then they coincide with the equations (14), (15) of Ref. [21] where it has been performed the substitution $\chi \longrightarrow \chi/a^2$. In that paper the equations have been solved by a cumbersome approximation method. It consists in a suitable transformation of the functions f, g by which the equation can be disentangled and finally reduced, for large s, to confluent hypergeometric equations. The energy spectrum is then obtained by truncating the hypergeometric series solution by a Schrodinger like quantization requirement. Therefore, in the present case, the energy spectrum of the H-atom with the "curved" Coulomb potential approximation (36), follows from the substitution $\chi \longrightarrow \chi/a^2$ in equation (40) of Ref. [21]:

$$\frac{k}{\mu} = \left[1 + \left(\frac{2\frac{\chi}{a^2} + \sqrt{4\frac{\chi^2}{a^4} + [(1+2n+2\gamma)^2 - 1][4\frac{\chi}{a^2}^2 + (1+2n+2\gamma)^2]}}{(1+2n+2\gamma)^2 - 1}\right)^2\right]^{-\frac{1}{2}} \tag{41}$$

It has been set $\gamma = \sqrt{\lambda^2 - (\chi/a^2)^2}$ that simplifies the mentioned calculations procedure

The above are two cases that have been approximately solved under the special condition (29). It seems however, on account of the arbitrariness of the functions E, M, that the "curved Coulomb" potential could assume very unusual expressions to make questionable the existence of solutions of the Dirac equation admitting an H-atom interpretation.

5 Remarks and comments

In previous calculations the H-atom has been approximately described by the Dirac equation with curved Coulomb potential in a class of LemâitreTolman Bondi cosmological models. On account of the approximations done, the cosmological model corresponds to a space time filled with spatially uniform distribution of dust matter so slowly varying to be considered constant in atomic time intervals.

The calculations are developed in a parameter dependent cosmological model satisfying conditions (26), (29). They are developed for two different approximations of the "curved" Coulomb potential.

The first is the basic one, namely the usual conventional Coulomb potential. The energy spectrum of the H-atom is explicitly determined. It coincides with that of the Minkowski space time having the Coulomb interacting constant that depends on the cosmological parameter.

The second approximation amounts to the basic Coulomb potential plus a linear term. The study is performed, in the same cosmological context of the previous one, and the energy levels are determined. In both cases the energy levels of the spectrum are determined exactly and have the form of those previously calculated for the Robertson Walker space time apart a dependence of the interacting constant by the cosmological parameter. The results have been obtained exactly, and not perturbatively.
In the special cosmological context assumed, it remains open the problem whether, under assumptions (26), (29), the H-atom exists also for the complete analytical expression of the Coulomb potential.

More generally, there is a wide class of LTB cosmologies, the functions E, M being arbitrary. The question arises then when the H-atoms do indeed exist. Even if the "curved" Coulomb potential is always attractive, the expansion of the Universe could be dominant.

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