Advanced Studies in Theoretical Physics
Vol. 15, 2021, no. 1, 37 - 44
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/astp.2021.91512

Multi-Step Amplifier for Very Small Wave Propagating on Cords

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Abstract

In a previous paper, it was found that a two-dimensional wave equation for a cord (thin rope) describing waves on a cord under the non-gravitational field reduces to a ordinary linear wave equation under the micro-amplitude approximation. And, the wave equation was denoted to have a unimodal solitary wave solution. In this paper, two cords are connected to one cord and all of them have the same linear density. Moreover, edges of the former cords are fixed to a bar which is swung to induce unimodal solitary waves propagating on them at the same time. There occur two reflective waves and a transmitted wave corresponding to the incident waves against a boundary among three cords. It is denoted that an amplification factor of the transmitted wave amplitude becomes 4/3 times. This is only one-step amplifier. A n-step amplifier makes the amplification factor to be $(4/3)^n$.

Keywords: wave equation, unimodal solitary wave, transmitted wave, n-step amplifier, amplification factor

1 Introduction

In 2019, we derived a two-dimensional wave equation for a cord (thin rope) under the non-gravitational field. We also obtained its unimodal solitary wave solution propagating on a cord with a sufficiently small amplitude.[2] There, we made it clear that the wave equation for a cord reduces to an ordinary linear wave equation under the micro-amplitude approximation. In general, the wave equation has the d'Lambert's solution.[1] Then, we found that two unimodal or kink solitary waves propagating in the opposite directions are stable against a collision.[2],[3]

In 2020, we have pointed out that the cord wave equation system and the string vibration system have many similar points in the sense that the same wave equations of them are derived under the micro-amplitude approximation and have multi-order modes of stationary sine waves.[4] In a previous paper, we have suggested cosmic experimental plans in International Space Station (ISS) to demonstrate a collision of two solitary waves under non-gravity. [5]

The aim of the present paper is to study the n-step amplifier from 2^n incident solitary waves to one tansmitted solitary wave. A 1-step amplifier is composed of two codes connected to one cord and all of them have the same linear density. The edges of the former cords are fixed to the bar which is swung to induce incident solitary waves at the same time propagating on them. Then, there appears an amplified transmitted solitary wave on the latter cord. Brief outline of this paper is denoted as follows. In Sect.2, we present the cord wave equation for the wave solutions propagating on the cord under the non-gravitational field. There, we point out that it reduces to a wave equation under the micro-amplitude approximation. In Sect.3.1, we take up the unimodal solitary wave solution of the wave equation as the incident wave expression. And, we derive the amplification factor of the transmitted wave. In Sect.3.2, we study the n-step amplifier. And concretely, we derive each amplification factor of 1, 2, 3-step amplifier. In Sect.3.3, we denote a numerical example of the amplified unimodal solitary waves. The last section is devoted to discussion. There, we investigate a problem that there occurs an inverse phase between the incident wave and the reflective wave. [7]

2 Wave Equation for Cord and Its Solitary Wave Solution

First, we assume that no external forces such as gravity are exerted on a cord and that stretching and contraction of a cord are negligible. And, we let a cord lie along the x-axis and a wave propagate on the xy-plane.

Next, we study the wave equation for a cord. The equation of motion for a cord reads

$$\sigma \frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial s}\left(T \frac{\partial z}{\partial s}\right), \tag{2.1}$$

where $z = x + iy$, t is time, s an arclength along a cord, σ a linear density of the cord, and T a norm of the tension vector. Here, we should remark that

$$z_s = \mathrm{e}^{i\theta}, \tag{2.2}$$

$$x_s^2 + y_s^2 = 1, \tag{2.3}$$

where $(\cos\theta, \sin\theta)$ is a unit tangential vector, and subscript s denotes partial differentiation with respect to s here and hereafter. Next, we shall introduce the following theorem.[2]

Theorem 1 *T is constant when there exists such a form of a solution as $z(s,t) = z(\xi) = z(s - ct)$, where c is an arbitrary constant.*

Then, Eq.(2.1) reduces to

$$\frac{\partial^2 z}{\partial t^2} - c^2 \frac{\partial^2 z}{\partial s^2} = 0, \tag{2.4}$$

where $c^2 = T/\sigma$. Under the micro-amplitude approximation, $x = s$ holds.[2] And, Eq. (2.4) becomes an ordinary linear wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{2.5}$$

where u is a displacement of a cord from the x-axis instead of y. This ordinary linear wave equation means that under the micro-amplitude approximation the present physical system is the linear wave one and it has the d'Alembert's solution, $u = h(x - ct) + k(x + ct)$.

Then, the sufficiently small unimodal solitary wave solution reads [2]

$$u = \delta \mathrm{sech}(x - x_0 - ct) \quad (\delta \approx 0). \tag{2.6}$$

3 Multi-Step Amplifier and Transmitted Wave

Now, we let two cords with the same linear density σ be connected to one cord via a joint with negligible mass and lie along the x-axis. The incident waves propagate rightward on the former cords at the same time by the bar to which their edges are fixed and the amplified transmitted wave propagates rightward on the latter cord with the same linear density σ. This is a 1-step amplifier system. In Fig.1, we show a rough sketch of this system.

3.1 1-step amplifier and transmitted wave

Here, from Eq.(2.6), we consider an incident unimodal solitary wave solution u_0:

$$u_0 = \delta \mathrm{sech}(x - ct) \quad (\delta \approx 0), \tag{3.1}$$

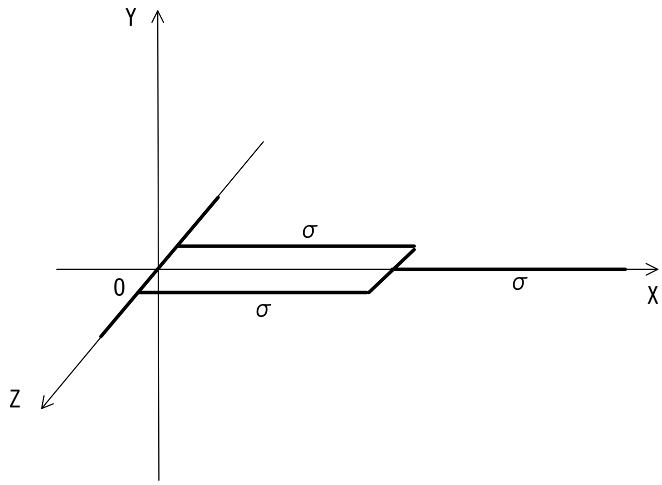

Figure 1: Rough sketch of 1-step amplifier.

and reflective and transmitted unimodal solitary wave solutions u_1 and u_2 and their velocity c:

$$u_1 = k_1 \delta \mathrm{sech}(x + ct), \tag{3.2}$$

$$u_2 = k_2 \delta \mathrm{sech}(x - ct), \tag{3.3}$$

$$c^2 = T/\sigma, \tag{3.4}$$

respectively.

The energy flow $v\mathcal{E}$ is expressed in the form: [8], [6]

$$v\mathcal{E} = -T \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}, \tag{3.5}$$

where $\mathcal{E}$ is an energy density. Then, the energy E transmitted by the wave u becomes

$$E = \int_{-\infty}^{\infty} v\mathcal{E} \mathrm{dx} = -T \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \mathrm{dx}. \tag{3.6}$$

Substituting Eq.(3.1) into Eq.(3.6), the energy E_0 of the incident wave is expressed as

$$E_0 = T\delta^2 c \int_{-\infty}^{\infty} \tanh^2(x - ct) \mathrm{sech}^2(x - ct) \mathrm{dx} = \frac{2}{3} T \delta^2 c. \tag{3.7}$$

Similarly, for the energies E_1 and E_2 of the reflective and transmitted waves, respectively, we get

$$E_1 = \frac{2}{3} T {k_1}^2 \delta^2 c, \tag{3.8}$$

$$E_2 = \frac{2}{3} T {k_2}^2 \delta^2 c. \tag{3.9}$$

Due to the energy conservation law $E_0+E_0 = E_1+E_1+E_2$ with Eqs.(3.7)-(3.9), we obtain

$$2 = 2{k_1}^2 + {k_2}^2. \tag{3.10}$$

In the same manner, the expression of the momentum p_0 transmitted by the incident wave reads [6]

$$p_0 = \sigma \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \mathrm{dx} = 2c\sigma\delta. \tag{3.11}$$

Similarly, p_1 and p_2 of the reflective and transmitted waves, respectively, become

$$p_1 = 2c\sigma k_1 \delta, \tag{3.12}$$

$$p_2 = 2c\sigma k_2 \delta. \tag{3.13}$$

Due to the momentum conservation law $p_0+p_0 = p_2-p_1-p_1$ with Eqs.(3.11)-(3.13), we obtain

$$2 = k_2 - 2k_1. \tag{3.14}$$

Eliminating k_2 from Eqs.(3.10) and (3.14), we have

$$k_1 = -\frac{1}{3}, \quad \text{or} \quad -1. \tag{3.15}$$

In the above result, the latter is an invalid solution, so that we have

$$k_1 = -\frac{1}{3}. \tag{3.16}$$

Eqs.(3.16) and (3.14) yield

$$k_2 = \frac{4}{3}. \tag{3.17}$$

Accordingly, the amplification factor of the 1-step amplifier becomes 4/3. Next, we shall derive expressions of the reflectivity R_f and the transmissivity T_r. The reflectivity R_f becomes

$$R_f = 2E_1/2E_0 = {k_1}^2 = \frac{1}{9}. \tag{3.18}$$

Similarly, the transmissivity T_r becomes

$$T_r = E_2/2E_0 = \frac{{k_2}^2}{2} = \frac{8}{9}. \tag{3.19}$$

From Eq.(3.16), we find that negative k_1 yields an inverse phase of the reflective waves.

3.2 N-step amplifier

In this subsection, we study the n-step amplifier which is composed of the n connected 1-step amplifiers. In Fig.2, we show a rough sketch of the 2-step amplifier. The transmitted wave becomes 4/3 times of the incident wave in each 1-step amplifier. Hence, the amplification factor of the n-step amplifier becomes $(4/3)^n$. Concretely, each amplification factor of 1, 2, 3-step amplifier is about 1.33, 1.78, 2.37 times, respectively.

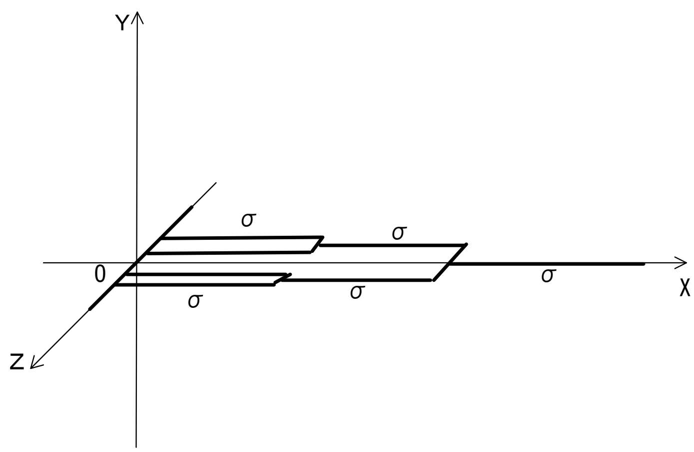

Figure 2: Rough sketch of 2-step amplifier.

3.3 Numerical example of amplified transmitted waves

Here, we shall present numerical example for the very small incident unimodal solitary waves and the amplified transmitted wave. We consider a case for $\delta = 0.007$, and $k_2 = 4/3$ for the 1-step amplifier. In this case, the reflectivity $R_f = 0.111$ (11.1%) and the transmissivity $T_r = 0.889$ (88.9%). In Fig.2, we plot the curves of the incident wave, and the amplified transmitted waves when 1, 2, 3-step amplifier. There, we find that the very small incident wave becomes detectable.

4 Discussion

In the present physical system, we have made sure that there occur the reflections and the amplified transmission for the incident waves at the boundary of two and one cords with linear density σ. Owing to the result such as the occurrence of the reflections and the transmission of waves under the

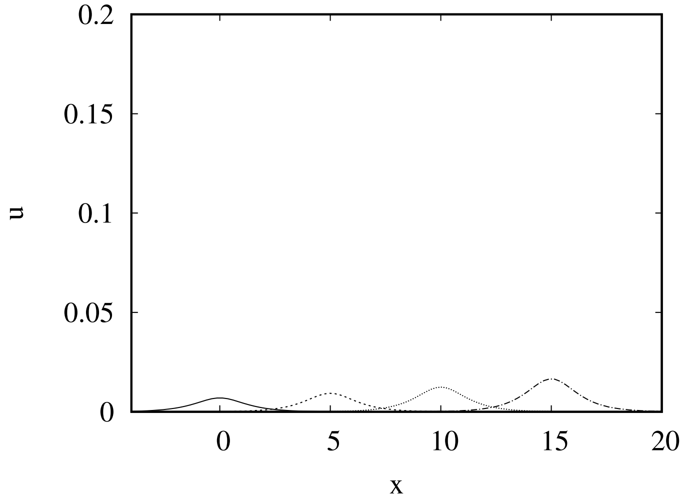

Figure 3: Incident wave and amplified transmitted waves.

non-gravitational field, the analysis in this paper may be useful for students to learn the wave theory. When $2\sigma > \sigma$, there occurs the inverse phase. This fact seems strange compared with the reflection of a light wave at a boundary of different density media. We explain this problem in appendix A.[7] At the end of this paper, we propose a cosmic experimental plan that we demonstrate the reflections and the amplified transmission phenomena in International Space Station (ISS) under non-gravity.

A Connected two cords with different linear densities

In Ref.[6], we have found that there occurs the inverse phase between the incident wave and the reflective wave at the boundary of the cords with the different linear densities σ_1(incident) $> \sigma_2$(transmitted). This fact is different from the case of electromagnetic waves. We try to explain this reason.[7] First, we shall introduce two impedances η_1 (incident) and η_2 (transmitted), and a reflective coefficient R_f and a transmission coefficient T_r as follows

$$R_f = (\eta_2 - \eta_1)/(\eta_1 + \eta_2), \quad T_r = 2\eta_2/(\eta_1 + \eta_2), \tag{A.1}$$

where $\eta = \sqrt{\mu/\epsilon} \propto \sqrt{1/\epsilon}$. Hence in case of light waves, when the medium density is large, the impedance η_1 becomes small and $R_f > 0$ (the identical phase). Here, the intensity I of a plane wave is defined as $I = |E|^2/(2\eta) \propto \int v\mathcal{E}dx = (2/3)T\delta^2\alpha$.[6] And, taking into account $|E|^2 \propto \delta^2$ and $\alpha = \sqrt{T/\sigma}$,[6] we have $\eta \propto \sqrt{\sigma}$. Accordingly in case of cord waves, when the linear density

σ_1 is large, the impedance η_1 becomes large and $R_f < 0$ (the inverse phase). Regarding the phase shift between the incident wave and the reflective wave, thus we have been able to explain the reason of the opposite phenomena here between electromagnetic waves and cord waves.

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Received: January 19, 2021; Published: January 30, 2021