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Fundamental Units and Constants of Physics

Richard P. Gilliard

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Abstract

It is generally believed that the most fundamental equations of physics require at least three units (length, mass, and time) and three fundamental constants (speed of light, gravitational constant, and Planck's constant). In this work it is demonstrated that these equations can be rewritten such that the speed of light c and the unit of time t appear only as the product ct. As a result, the fundamental equations can be written in four dimensional format using only length and mass as units and with just two new constants (containing only those units) replacing the usual three. In addition Planck length and Planck mass are shown to be simple combinations of these two new constants. Similarly, the fine-structure constant is defined using these two constants instead of the usual three. The speed of light is only needed as the linear scale factor for the final conversion of fourth dimensional length to time.

Keywords: fundamental constants, speed of light, general relativity, quantum mechanics

1 Introduction

Relativistic mechanics demonstrates the interrelationships and apparent equivalences between space and time. One would therefore expect that, if the fundamental equations of physics were written in four dimensional form (utilizing length as the unit for all dimensions), the unit of time and all factors containing that unit would be eliminated. However, at least two of the three fundamental equations (Maxwell's equations of electromagnetism and Einstein's equations for General Relativity) contain the factor c, the speed of light, in them regardless of the system of units. The General Relativity equations also contain the gravitational constant G, which also involves the unit of time. Leaving these equations in four dimensional form does not eliminate these complications, and to my knowledge it has never been demonstrated that it is possible to reformat them in a way to do so. The other fundamental equation, the Schrödinger equation, contains Planck's constant $\hbar$, which also contains the unit of time. In this work, after introducing some natural new definitions, it will be shown that the unit time and the speed of light c in fact need only appear in the equations as the product ct, showing that c and t can be mathematically separated out of the fundamental equations of physics, using instead the fourth dimensional length component. These changes also result in c being eliminated as a constituent of the other fundamental constants of physics except Planck time which is simply equal to Planck length divided by c. The physical significance of these facts will also be discussed.

2 Basic Considerations

It is well known that electromagnetic quantities to include charge, electromagnetic potential, and electric and magnetic field can be expressed in Gaussian cgs units using only cm, gm, and sec (length, mass, and time) as units **[1]**. As a result, the starting point here will be the appropriate cgs units for these quantities rather than for example the statcoulomb or statvolt. Table 1 shows the definition of appropriate quantities both ways. The primary insight and motivation for this work is the realization that in four dimensional format, the natural unit for kinetic energy is mass rather than energy (gram rather than erg in cgs units). In standard relativistic mechanics in fact velocity is usually expressed as a dimensionless quantity since the derivative of a space dimension (length) with respect to the fourth dimension is also without units as shown in Landau and Lifshitz, vol 2. **[2]** Eq. (7.1).

$$u^i = \frac{dx^i}{ds} \tag{1}$$

All of the physical content is present in this expression that is present in the standard definition of velocity. Only the factor c is absent. The consequence of this fact is that kinetic energy using the dimensionless definition of velocity has the unit of mass. This insight in turn motivates examining the effect of using factors of c in the equations to alter the units of energy in certain expressions without changing their fundamental physical content.

3 Conversion of Units

Where applicable, in all that follows, the four dimensional format of Landau and Lifshitz **[3]** will be used throughout. Landau and Lifshitz, vol 2. **[4]** Eq. (16.1), expressing the action of a particle in an electromagnetic field, is repeated below as Eq. (2).

$$S = \int_a^b \left(-mcds - \frac{e}{c}A_i dx^i\right) \tag{2}$$

The parameters are defined as follows:

- S = the action function
- m = particle mass
- c = the speed of light
- s = the four dimensional interval over which the Lagrangian (the integrand) is integrated (i.e. from a to b)
- e = particle charge
- A_i = the covariant components of the electromagnetic potential four-vector
- x^i = the contravariant components of the four-space vector

The Einstein summation convention is used in all equations with repeated indices. As noted by Landau and Lifshitz **[5]**, “It should be pointed out that, so long as we have no formulas relating the charge or potentials with already known quantities, the units for measuring these new quantities can be chosen arbitrarily.” In this equation (Gaussian cgs units) it is clear that the product of charge and the potential (eA_i) has been chosen to have the units of energy. It has already been shown that there is no loss of physical content with regard to kinetic energy by leaving velocity in its natural four dimensional format (mathematically equivalent to dividing it by c^2). In this work, the unit of charge will thus be redefined by dividing the statcoulomb by c as will the potential by dividing the statvolt by c. Finally, wherever the unit of energy appears in a quantity, that quantity will be redefined by dividing it by c^2 in order to make it dimensionally consistent with four dimensional kinetic energy. These redefinitions of units for the physical quantities are shown in Table 1. Again the physical content inherent in a quantity is not altered by multiplying or dividing it by a constant.
The appropriate starting point is the three sets of equations already identified as the fundamental equations of theoretical physics in their standard Gaussian cgs form. Again in all cases we follow the format and notation of Landau and Lifshitz. Maxwell’s equations (26.1), (26.2), (30.3), and (30.4) from Landau and Lifshitz vol. 2 **[6]** are repeated below.

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \tag{3}$$

$$\nabla \cdot \boldsymbol{H} = 0 \tag{4}$$

$$\nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j} \tag{5}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \tag{6}$$

The $\boldsymbol{j}$ vector has for its x, y, and z components

$$j^i = \rho \frac{dx^i}{dt} \tag{7}$$

Einstein's equation of General Relativity (95.5) from Landau and Lifshitz vol. 2 **[7]** is repeated below. Note that the more commonly used symbol G has been substituted for the k that appears in the reference.

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik} \tag{8}$$

The Schrödinger equation (17.6) from Landau and Lifshitz vol. 3 **[8]** is repeated below.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right) \nabla^2 \Psi + U(x, y, z)\Psi \tag{9}$$

The equations will now be rewritten reflecting only the changes required to represent the new units of the quantities shown in Table 1 by algebraically replacing the old values of $\boldsymbol{E}, \boldsymbol{H}$, ρ, $\boldsymbol{j}$, T_{ik}, and $U(x, y, z)$ with the new.

Gaussian cgs Quantity	Quantity Name	Appears in	Units of Gaussian cgs Quantity	Units of New Quantity	Conversion Factor
e	Charge	Electromagnetic Derivation	$statC = cm^{1.5} gm^{0.5} sec^{-1}$	$cm^{0.5}\ gm^{0.5}$	$old = new \times c$
A	Four Potential	Electromagnetic Derivation	$statV = cm^{0.5} gm^{0.5} sec^{-1}$	$cm^{-0.5}\ gm^{0.5}$	$old = new \times c$
E	Electric Field	Maxwell's Equations	$\frac{statV}{cm} = cm^{-0.5} gm^{0.5} sec^{-1}$	$cm^{-1.5}\ gm^{0.5}$	$old = new \times c$
H	Magnetic Field	Maxwell's Equations	$Oe = cm^{-0.5} gm^{0.5}\ sec^{-1}$	$cm^{-1.5}\ gm^{0.5}$	$old = new \times c$
ρ	Electric Charge Density	Maxwell's Equations	$\frac{statC}{cm^3} = cm^{-1.5} gm^{0.5} sec^{-1}$	$cm^{-2.5}\ gm^{0.5}$	$old = new \times c$
j	Electric Current Density	Maxwell's Equations	$\frac{statC}{cm^2 sec} = cm^{-0.5} gm^{0.5} sec^{-2}$	$cm^{-2.5}\ gm^{0.5}$	$old = new \times c^2$
T	Stress Energy Tensor	Einstein Equation	$\frac{erg}{cm^3} = cm^{-1} gm\ sec^{-2}$	$cm^{-3} gm$	$old = new \times c^2$
U	Potential Energy	Schrödinger Equation	$erg = cm^2 gm\ sec^{-2}$	gm	$old = new \times c^2$

Table 1. Conversion factors from Gaussian cgs units to new units

Note that this includes the change in definition of the components of j^i.

$$\nabla \times c\boldsymbol{E} = -\frac{1}{c}\frac{\partial c\boldsymbol{H}}{\partial t} \tag{10}$$

$$\nabla \cdot c\boldsymbol{H} = 0 \tag{11}$$

$$\nabla \times c\boldsymbol{H} = \frac{1}{c}\frac{\partial c\boldsymbol{E}}{\partial t} + \frac{4\pi}{c} c^2 \boldsymbol{j} \tag{12}$$

$$\nabla \cdot c\boldsymbol{E} = 4\pi c\rho \tag{13}$$

$$j^i = \rho \frac{dx^i}{dx^0} \tag{14}$$

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} c^2 T_{ik} \tag{15}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right) \nabla^2 \Psi + c^2 U(x,y,z)\Psi \tag{16}$$

In Table 2, the definitions of two new constants are introduced. They can be roughly thought of as simplified replacements for the gravitational constant and Planck's constant (γ for G and η for ħ), but each having only the units of length and mass unlike the constants they replace. Conversion factors are shown in Table 2.

Old Constant	Appears in	Value of Old Constant	New Constant	Value of New Constant	Substitution
G	Einstein Equation	6.6741×10^{-8} cm^3/gm-sec^2	γ	7.4259×10^{-29} cm/gm	$G = \gamma c^2$
$\hbar$	Schrödinger Equation	1.0546×10^{-27} cm^2 gm/sec	η	3.5177×10^{-38} gm-cm	$\hbar = \eta c$

Table 2. Definition of new constants

The equations are then further rewritten to reflect direct substitution of expressions containing the new constants for G and ħ and changing t to $\frac{1}{c}x^0$.

$$\nabla \times c\boldsymbol{E} = -\frac{1}{c}\frac{\partial c\boldsymbol{H}}{\frac{1}{c}\partial x^0} \tag{17}$$

$$\nabla \cdot c\boldsymbol{H} = 0 \tag{18}$$

$$\nabla \times c\boldsymbol{H} = \frac{1}{c}\frac{\partial c\boldsymbol{E}}{\frac{1}{c}\partial x^0} + \frac{4\pi}{c} c^2 \boldsymbol{j} \tag{19}$$

$$\nabla \cdot c\boldsymbol{E} = 4\pi c\rho \tag{20}$$

$$j^i = \rho \frac{dx^i}{dx^0} \tag{21}$$

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi \gamma c^2}{c^4} c^2 T_{ik} \tag{22}$$

$$i\eta c \frac{\partial \Psi}{\frac{1}{c}\partial x^0} = -\left(\frac{\eta^2 c^2}{2m}\right) \nabla^2 \Psi + c^2 U(x^1, x^2, x^3)\Psi \tag{23}$$

Finally, the equations are rationalized (dividing through by c or c^2 where appropriate).

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{H}}{\partial x^0} \tag{24}$$

$$\nabla \cdot \boldsymbol{H} = 0 \tag{25}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{E}}{\partial x^0} + 4\pi \boldsymbol{j} \tag{26}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \tag{27}$$

$$j^i = \rho \frac{dx^i}{dx^0} \tag{28}$$

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi\gamma T_{ik} \tag{29}$$

$$i\eta \frac{\partial \Psi}{\partial x^0} = -\left(\frac{\eta^2}{2m}\right) \nabla^2 \Psi + U(x^1, x^2, x^3)\Psi \tag{30}$$

Thus the fundamental equations have been rewritten in four dimensional format (i.e. not using the unit of time) with only two fundamental constants each having the units of only length and mass. These constants are the following:

$$\gamma = 7.4259 \text{ x } 10^{-29} \text{ cm/gm} \tag{31}$$

$$\eta = 3.5177 \text{ x } 10^{-38} \text{ gm-cm} \tag{32}$$

Note also below that, by substituting from Table 2, Planck length l_p and Planck mass m_p **[9]** are obtained as simple combinations of γ and η. Planck time (equal to Planck length divided by c) naturally requires the use of c, for the final conversion.

$$\sqrt{\gamma\eta} = \sqrt{\frac{G}{c^2}\frac{\hbar}{c}} = \sqrt{\frac{G\hbar}{c^3}} = l_p = 1.6162 \text{ x } 10^{-33} \text{ cm} \tag{33}$$

$$\sqrt{\frac{\eta}{\gamma}} = \sqrt{\frac{\hbar}{c}\frac{c^2}{G}} = \sqrt{\frac{\hbar c}{G}} = m_p = 2.1765 \text{ x } 10^{-5} \text{ gm} \tag{34}$$

The value of the elementary charge in Gaussian units is 4.8032 x 10^{-10} statcoulomb. Therefore by Table 1 the value of elementary charge in the new units is this value in Gaussian units divided by the speed of light.

$$q_e = 1.6021\ x\ 10^{-20}\ (\text{cm-gm})^{0.5} \tag{35}$$

Note that there is a dimensionless number that can be formed from η and q_e. That number is $\frac{\eta}{q_e{}^2}$ or its inverse. Calculating,

$$\frac{\eta}{q_e{}^2} = 137.0 = \alpha^{-1} \tag{36}$$

Here α is identically equal to the fine-structure constant used in quantum electrodynamics **[10]**. Direct substitutions from Table 2 verify the exact equivalence of this expression to the standard definition of α. Thus, the fine-structure constant is simply the ratio of the new quantum constant η and the square of the elementary charge, neither of which involve *c* nor the unit of time.
Note also that Eq. (33) and (34) can be algebraically rearranged as follows:

$$\eta = m_p\, l_p \tag{37}$$

$$\gamma = \frac{m_p}{l_p} \tag{38}$$

This means that the fundamental equations can be rewritten with only l_p and m_p (Planck length and Planck mass) as constants. Maxwell's equations Eq. (24) to Eq. (27) are unchanged with all constants already eliminated. Eq. (29) and Eq. (30) become the following as shown below.
Einstein:

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi \left(\frac{m_p}{l_p}\right) T_{ik} \tag{39}$$

Schrödinger:

$$i m_p\, l_p \frac{\partial \Psi}{\partial x^0} = -\left(\frac{m_p{}^2 l_p{}^2}{2m}\right) \nabla^2 \Psi + U(x^1, x^2, x^3)\Psi \tag{40}$$

Similarly, Eq. (36) can be rewritten, expressing the fine-structure constant only in terms of Planck mass and Planck length and electronic charge.

$$\frac{\eta}{q_e{}^2} = \frac{m_p\, l_p}{q_e{}^2} = 137.0 = \alpha^{-1} \tag{41}$$

4 Results and Conclusions

The first result is that the fundamental equations and constants of theoretical physics have been rewritten without loss of generality or rigor using only two units and two fundamental constants, a simplification from the three of each normally deemed necessary. The constant c is left only to make the final conversion from the fourth dimensional length to time, but not needed for any other calculation or expression. It should be stated that the analysis would result in the same conclusion if the two fundamental units were chosen as mass and time rather than mass and length; however, the required number of units and constants could not be reduced to two each if mass were not included within one of the units. The other key result is that the other key parameters of theoretical physics (Planck length, Planck mass, and the fine-structure constant) require only these two constants rather than the usual three for their calculation. Planck time (equal to Planck length divided by c), of course, cannot be written without the use of c, which again underscores that ultimately the only need for c in theoretical physics is to convert the fourth dimensional length to time.

The importance of this result is that it provokes thought in a number of areas, including the following:

- The speed of light has always been thought of as a basic constant of physics. To what extent is this so, and to what extent is it a constant which expresses the way we have evolved to experience and measure the fourth dimension?
- Ultimately how many units, constants, symmetries, dimensions, etc. are required to express physical law? This is, of course, being addressed in string theory among others.
- Perhaps most importantly, there are many systems for expressing electromagnetic units which include many different constants and arcane relationships among the systems and even among the units within the systems. Would physics and engineering be better served by a simpler more elegant system which better clarifies underlying physical and mathematical relationships, such as that put forward in this work?

References

[1] Barton Zwiebach, *A First Course in String Theory*, Second Edition, Cambridge University Press, Cambridge, 2009, p. 14.
https://doi.org/10.1017/cbo9780511841620.003

[2] Lev D. Landau and Evgeny M. Lifshitz, *The Classical Theory of Fields*, Third English Edition, Pergamon Press Ltd., Oxford, 1971, p. 21.

[3] Ibid., p. xi.

[4] Ibid., p. 45.

[5] Ibid.

[6] Ibid., pp. 66, 74.

[7] Ibid., p. 274.

[8] Lev D. Landau and Evgeny M. Lifshitz, *Quantum Mechanics*, Second English Edition, Pergamon Press Ltd., Oxford, 1965, p. 51.

[9] Zwiebach, A *First Course in String Theory*, p. 61.

[10] Ibid., p. 62.

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