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How to Commute

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Abstract

A simple exposition of the rarely discussed fact that a set of free boson fields describing different, i.e. kinematically different particle types can be quantized with mutual anticommutation relations is given by the explicit construction of the Klein transformations changing anticommutation relations into commutation relations. The q-analog of the presented results is also treated. The analogous situation for two independent free fermion fields with mutual commutation or anticommutation relations is briefly investigated.

Keywords: canonical quantization, spin and statistics, quantum field theory

1 Introduction

All hitherto existing experimental evidence indicates that physical systems with one type of integer spin particles solely obey the laws of Bose-Einstein statistics, whereas systems with one type of half-odd integer spin particles respect Fermi-Dirac statistics. The natural way to arrive at Bose-Einstein or Fermi-Dirac statistics is to describe the particles by the help of quantum fields which commute or anticommute for space-like separations, repectively. When one turns from the commutation relations for a given field to those between different fields in the sense that the fields cannot be mapped by space-time transformations onto each other, the situation becomes more complicated. One observes that 'abnormal' commutation relations in theories in which, e.g., two different integer spin fields anticommute, may arise, but such theories possess special symmetries which allow to link them to the case with regular commutation relations. In this paper, it is shown how this link can easily be constructed from simple algebraic considerations for systems with a finite number of degrees of freedom which can be generalized in a straightforward manner to the case of inifinitely many degrees of freedom.

Starting from the well-known commutation relations of the rising and lowering operators of two independent, i.e. non-interacting bosonic harmonic oscillators

$$[a, a^\dagger] = 1\,, \quad [b, b^\dagger] = 1\,,$$

where a and b annihilate the ground state

$$a|0\rangle = b|0\rangle = 0\,, \tag{1}$$

one may impose 'abnormal' mutual anticommutation relations given by

$$\{a, b\} = ab + ba = \{a, b^\dagger\} = 0\,, \tag{2}$$

consequently leading to the Hermitian conjugate relations

$$\{a^\dagger, b^\dagger\} = \{a, b\}^\dagger = \{a, b^\dagger\}^\dagger = \{a^\dagger, b\} = 0\,. \tag{3}$$

The commutation relations given by eq. (1) fix the physical nature of the phonons as bosons, since the creation operators $a^\dagger$ and $b^\dagger$ can create an arbitrary number of phonons, contrary to the fermionic case where the corresponding anticommutation relations would imply the Pauli principle $a^{\dagger\,2} = b^{\dagger\,2} = 0$. In the bosonic case, a state normalized to one containing $\tilde{n}$ phonons of type a would be given, e.g., by

$$|\tilde{n}\rangle = \frac{1}{\sqrt{\tilde{n}!}} a^{\dagger\,\tilde{n}}|0\rangle\,. \tag{4}$$

The anticommutation relations in eq. (2) can be converted into commutation relations by mapping the algebra of b-type operators only according to

$$a \mapsto \tilde{a} = a\,, \quad b \mapsto \tilde{b} = \eta b\,, \quad b^\dagger \mapsto \tilde{b}^\dagger = b^\dagger \eta = -\eta b^\dagger\,, \tag{5}$$

where the involutive, unitary and Hermitian phonon (or particle) number parity operator η is defined via the phonon number operator

$$n = a^\dagger a + b^\dagger b \tag{6}$$

by

$$\eta = (-1)^n = e^{i\pi n} = e^{-i\pi n} = \eta^{-1} = \eta^\dagger \,. \tag{7}$$

η anticommutes with a, $a^\dagger$, b, and $b^\dagger$, since the creation and annihilation operators change the particle number by one. Hence, the commutation relations in the b-sector are preserved, since $[\tilde{b}, \tilde{b}] = 0$ is trivially fulfilled and from $\eta^2 = 1$ follows

$$[\tilde{b}, \tilde{b}^\dagger] = [\eta b, b^\dagger \eta] = \eta b b^\dagger \eta - b^\dagger \eta^2 b = [b, b^\dagger] = 1 \,, \tag{8}$$

however one now has mutual commutativity

$$[a, \tilde{b}] = a\eta b - \eta b a = -\eta a b - \eta b a = -\eta \{a, b\} = 0 \,,$$
$$[a, \tilde{b}^\dagger] = a b^\dagger \eta - b^\dagger \eta a = \{a, b^\dagger\} \eta = 0 \,,$$

i.e. the anticommutation relations between the a- and b-operators go over into commutation relations by a change of phase conventions without changing the physical content of the theory. Note, however, that the transformation according to eq. (5) does not correspond to a unitary transformation of the operators and the underlying Hilbert space of phonon states which would preserve the commutation rules.

The way to achieve a situation where all operators fulfill standard commutation rules is, of course, not unique. E.g., introducing particle number and particle number parity operators for different phonon types

$$n_a = a^\dagger a \,, \quad n_b = b^\dagger b \,, \tag{9}$$

$$\eta_a = (-1)^{n_a} = \eta_a^{-1} = \eta_a^\dagger \,, \quad \eta_b = (-1)^{n_b} = \eta_b^{-1} = \eta_b^\dagger \tag{10}$$

and new operators

$$a \mapsto \tilde{a} = \eta_b a \,, \quad a^\dagger \mapsto \tilde{a}^\dagger = a^\dagger \eta_b = \eta_b a^\dagger \,, \quad b \mapsto \tilde{b} = \eta_b b \,, \quad b^\dagger \mapsto \tilde{b}^\dagger = b^\dagger \eta_b = -\eta_b b^\dagger \,, \tag{11}$$

also does the job. A further valid redefinition is given by

$$a \mapsto \tilde{a} = a \,, \quad b \mapsto \tilde{b} = \eta_a b \,, \quad b^\dagger \mapsto \tilde{b}^\dagger = b^\dagger \eta_a = \eta_a b^\dagger \,. \tag{12}$$

So-called Klein transformations as presented above have been introduced for the first time by Oskar Klein [1]. The abstract work on a quantum field theoretical level in connection with the spin-statistics theorem given much later by Huzihiro Araki [2] was the basis for a short discussion given by Ray Streater and Arthur Wightman in their famous book on PCT, spin and statistics, and all that [3].

2 Several degrees of freedom

In the case where m different phonon types are created by operators $a_1^\dagger, \ldots a_m^\dagger$ with

$$[a_i, a_i^\dagger] = 1\,, \quad \{a_i, a_j\} = \{a_i, a_j^\dagger\} = 0 \quad \text{for } i \neq j\,, \tag{13}$$

the mutual anticommutation relations can be successively transformed into commutation relations by the following sequence of transformations

$$\begin{aligned} a_1 &\mapsto \tilde{a}_1 = a_1\,, \\ a_2 &\mapsto \tilde{a}_2 = \eta_1 a_2\,, \\ a_3 &\mapsto \tilde{a}_2 = \eta_1 \eta_2 a_3\,, \\ &\ldots \\ a_i &\mapsto \tilde{a}_i = \eta_1 \cdot \ldots \cdot \eta_{i-1} a_i\,, \\ &\ldots \\ a_m &\mapsto \tilde{a}_m = \eta_1 \cdot \ldots \cdot \eta_{m-1} a_m\,, \end{aligned} \tag{14}$$

where

$$\eta_i = (-1)^{a_i^\dagger a_i}\,. \tag{15}$$

This explicit construction shows that there is always the possibility to successively remove all abnormal anticommutators from a theory describing different bosons only. An analogous statement holds for the general case involving different different bosonic and different fermionic particles. For the even more general quantum field theoretical case where one has infinitely many degrees of freedom, the (normal) abnormal case of two different Fermi fields with vanishing (anti-)commutators is briefly discussed in the last section. But before the situation discussed above shall be reanalyzed from a q-deformed point of view.

3 How to q-commute

Considering again the commutation relations of the raising and lowering operators of two independent bosonic harmonic oscillators

$$[a, a^\dagger] = 1, [b, b^\dagger] = 1, [a, b] = [a, b^\dagger] = [b^\dagger, a^\dagger] = [b, a^\dagger] = 0, \tag{16}$$

where a and b annihilate the ground state $a|0\rangle = b|0\rangle = 0$, one may introduce the map

$$\tilde{a} = a, \quad \tilde{b} = \eta b, \quad \tilde{a}^\dagger = a^\dagger, \quad \tilde{b}^\dagger = b^\dagger \eta^\dagger = b^\dagger \eta^{-1} = q\eta^{-1} b^\dagger \tag{17}$$

where η is a unitary q-parity operator defined with θ real as

$$\eta = q^N = e^{i\theta N}\,, \quad \eta^\dagger = e^{-i\theta N} = \eta^{-1}\,, \tag{18}$$

and

$$N = N_a + N_b = a^\dagger a + b^\dagger b\,. \tag{19}$$

Introducing the following commutation relations

$$\begin{aligned} \eta a &= \frac{1}{q} a\eta\,, \quad \eta a^\dagger = q a^\dagger \eta\,, \\ \eta b &= \frac{1}{q} b\eta\,, \quad \eta b^\dagger = q b^\dagger \eta\,, \end{aligned} \tag{20}$$

leads to the algebra

$$\begin{gathered} [\tilde{a}, \tilde{a}^\dagger] = 1\,, \quad [\tilde{b}, \tilde{b}^\dagger] = 1\,, \\ [\tilde{a}, \tilde{b}]_q = \tilde{a}\tilde{b} - q\tilde{b}\tilde{a} = 0\,, \\ [\tilde{a}, \tilde{b}^\dagger]_{\frac{1}{q}} = \tilde{a}\tilde{b}^\dagger - \frac{1}{q}\tilde{b}^\dagger\tilde{a} = 0\,, \\ [\tilde{N}_a, \tilde{N}_b] = 0\,, \end{gathered} \tag{21}$$

where

$$\tilde{N}_a = \tilde{a}^\dagger\tilde{a}, \quad \tilde{N}_b = \tilde{b}^\dagger\tilde{b}\,. \tag{22}$$

Replacing $\eta = q^N$ with $\eta_a = q^{N_a}$ in eq. (18) gives the same result. Indeed, the map given in eq. (17) transforms commuting modes into q-commuting modes without changing the boson algebra for each mode. For $q = -1$, the map eq. (17) leads to the case which has been introduced for the first time by Klein [1], and studied in further detail by Araki [2].

One may also define another mapping like

$$\tilde{a} = \eta_b a\,, \quad \tilde{b} = \eta_b b\,, \quad \tilde{a}^\dagger = a^\dagger \eta_b^\dagger\,, \quad \tilde{b}^\dagger = b^\dagger \eta_b^\dagger\,, \tag{23}$$

where $\eta_b = q^{N_b}$. Then we have the following commutation relations

$$\begin{gathered} [\tilde{a}\,, \tilde{a}^\dagger] = 1, \quad [\tilde{b}\,, \tilde{b}^\dagger] = 1\,, \\ [\tilde{a}\,, \tilde{b}]_{\frac{1}{q}} = 0, \quad [\tilde{a}\,, \tilde{b}^\dagger]_q = 0\,, \\ [\tilde{N}_a, \tilde{N}_b] = 0\,. \end{gathered} \tag{24}$$

These considerations can be generalized into the multi-mode case. Considering the n independent bosonic harmonic oscillators ($i \neq j$)

$$[a_i, a_i^\dagger] = 1, \quad i = 1, 2, 3, \ldots, n\,,$$

$$[a_i, a_j] = [a_i, a_j^\dagger] = 0\,, \tag{25}$$

leads to the consideration of the following map

$$\tilde{a}_i = \Lambda_{i-1} a_i, \quad \tilde{a}_i^\dagger = a_i^\dagger \Lambda_{i-1}^{-1}\,, \tag{26}$$

where Λ_{i-1} is a unitary q-parity operator defined as

$$\Lambda_{i-1} = \prod_{k=1}^{i-1} \eta_k = \prod_{k=1}^{i-1} q^{N_k} \tag{27}$$

and

$$N_k = a_k^\dagger a_k\,. \tag{28}$$

Then, one has the commutation relations

$$\begin{aligned} &[\tilde{a}_i, \tilde{a}_i^\dagger] = 1, \quad i = 1, 2, 3, \ldots, n\,, \\ &[\tilde{a}_i, \tilde{a}_j]_q = 0, \quad i < j\,, \\ &[\tilde{a}_i, \tilde{a}_j]_{\frac{1}{q}} = 0, \quad i > j\,, \\ &[\tilde{a}_i, \tilde{a}_j^\dagger]_{\frac{1}{q}} = 0, \quad i < j \\ &[\tilde{a}_i, \tilde{a}_j^\dagger]_q = 0, \quad i > j\,, \\ &[\tilde{N}_i, \tilde{N}_j] = 0\,, \end{aligned} \tag{29}$$

where

$$\tilde{N}_i = \tilde{a}_i^\dagger \tilde{a}_i\,. \tag{30}$$

The second, third, forth and fifth relations can be also written as

$$[\tilde{a}_i, \tilde{a}_j]_{q^{\epsilon_{ij}}} = 0, \quad [\tilde{a}_i, \tilde{a}_j^\dagger]_{q^{-\epsilon_{ij}}} = 0\,, \tag{31}$$

where

$$\epsilon_{ij} = \begin{cases} 1 & (i < j) \\ 0 & (i = j) \\ -1 & (i > j) \end{cases} \tag{32}$$

4 Changing the mutual commutation relations of two different Dirac fields

For the sake of generality, one may also have a look at the situation where fermion fields are involved. However, only a simple case involving free fields shall be discussed for the sake of brevity. Discussing free fields only is not a major disadvantage, since a rigorous construction of a non-trivial quntum field theory in four space-time dimensions has not been successful so far, and most of our practical knowledge in local quantum field theory is based on considerations concerning free fields acting as operator valued distributions on a Fock space.

A Dirac field describing non-interacting spin-$\frac{1}{2}$ fermions like, e.g., the free electron-positron field, can be written in the following form, using natural units $\hbar = c = 1$ and a relativistic notation with $kx = k_\mu x^\mu = k^0 x^0 - \vec{k}\vec{x}$, $k^0 = \sqrt{\vec{k}^2 + m_e^2} > 0$

$$\psi(x) = \int \frac{d^3k}{2k_0(2\pi)^3} \sum_{s=\pm\frac{1}{2}} \{e^{-ikx} u_s(\vec{k}) a_s(\vec{k}) + e^{+ikx} v_s(\vec{k}) b_s^\dagger(\vec{k})\} , \tag{33}$$

where the $u_s(\vec{k})$ [$v_s(\vec{k})$] denote electron [positron] spinors for the corresponding spin $s = \pm\frac{1}{2}$ and momentum $\vec{k} = (k^1, k^2, k^3)$. The electron [positron] destruction operators $a_s(\vec{k})$ [$b_s(\vec{k})$] and the Hermitian adjoint creation operators $a_s^\dagger(\vec{k})$ [$b_s^\dagger(\vec{k})$] fulfill the anti-commutation relations

$$\{a_s(\vec{k}), a_{s'}^\dagger(\vec{k}')\} = \{b_s(\vec{k}), b_{s'}^\dagger(\vec{k}')\} = \delta_{ss'} 2k^0 (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') ,$$

$$\{a_s(\vec{k}), a_{s'}(\vec{k}')\} = \{a_s^\dagger(\vec{k}), a_{s'}^\dagger(\vec{k}')\} = \{b_s(\vec{k}), b_{s'}(\vec{k}')\} = \{b_s^\dagger(\vec{k}), b_{s'}^\dagger(\vec{k}')\} = 0 ,$$

$$\{a_s(\vec{k}), b_{s'}(\vec{k}')\} = \{a_s^\dagger(\vec{k}), b_{s'}^\dagger(\vec{k}')\} = \{a_s(\vec{k}), b_{s'}^\dagger(\vec{k}')\} = \{a_s^\dagger(\vec{k}), b_{s'}(\vec{k}')\} = 0 . \tag{34}$$

Note that all the destruction operators annihilate the vacuum according to

$$a_s(\vec{k})|0\rangle = b_s(\vec{k})|0\rangle = 0 \tag{35}$$

in order to have a Fock Hilbert space representation of free electron and positron states. Considering an extended theory including a further type of kinematically independent fermions of mass m', e.g.muons, one introduces the additional 'primed' Dirac field

$$\psi'(x) = \int \frac{d^3k}{2k_0'(2\pi)^3} \sum_{s=\pm\frac{1}{2}} \{e^{-ikx} u_s'(\vec{k}) a_s'(\vec{k}) + e^{+ikx} v_s'(\vec{k}) {b_s'}^\dagger(\vec{k})\} . \tag{36}$$

with $k_0' = \sqrt{\vec{k}^2 + m'^2}$ and creation and destruction operators fulfilling completely analogous anti-commutation relations as given by eqns. (34).

It is common practice to assume that the creation and destruction operators for different fermion types which are (kinematically) independent in the sense that they cannot be transformed by a Poicaré transformations or C, P, or T into each other [4] *anticommute*, i.e. one has

$$\{\hat{c}, \hat{c}'\} = 0\,, \tag{37}$$

where $\hat{c}$ represents any creation or destruction operator for particles of mass m appearing in eqns. (34) and $\hat{c}'$ a corresponding creation or destruction operator for particles of mass m'.

Introducing the total particle number operator

$$N = N_a + N_b + N_{a'} + N_{b'} =$$
$$\sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{2k^0(2\pi)^3} \Big[a_s^\dagger(\vec{k}) a_s(\vec{k}) + b_s^\dagger(\vec{k}) b_s(\vec{k}) + {a'_s}^\dagger(\vec{k}) a'_s(\vec{k}) + {b'_s}^\dagger(\vec{k}) b'_s(\vec{k}) \Big] \tag{38}$$

one may define the operator

$$\eta = (-1)^N = e^{i\pi N}\,. \tag{39}$$

η is defined on the whole Fock Hilbert space and fulfills

$$\eta = \eta^\dagger\,, \quad \eta^2 = 1\,, \quad \eta^{-1} = \eta^\dagger\,. \tag{40}$$

Note that η could also be defined by the help of the charge operator

$$\eta = e^{i\pi Q}\,, \quad Q = -N_a + N_b - N_{a'} + N_{b'}\,. \tag{41}$$

Contrary to the particle number operator N, Q is conserved when interactions are involved and therefore provides some advantages when one tries to discuss an interacting theory, where the charge structure survives rather than the particle picture. Since creation and destruction operators change the particle number by ± 1, one has

$$\hat{c}\eta = -\eta\hat{c}\,, \ \text{ or } \ \{\eta, \hat{c}\} = 0\,, \tag{42}$$

and $\{\eta, \hat{c}'\} = 0$. Now one may define a new algebra of creation and destruction operators for particles of mass m', explicitly

$$a'_s(\vec{k}) \mapsto \tilde{a}_s(\vec{k}) = \eta a'_s(\vec{k})\,, \quad b'_s(\vec{k}) \mapsto \tilde{b}_s(\vec{k}) = \eta b'_s(\vec{k})\,, \tag{43}$$

implying

$${a'_s}^\dagger(\vec{k}) \mapsto \tilde{a}_s^\dagger(\vec{k}) = {a'_s}^\dagger(\vec{k})\eta\,, \quad {b'_s}^\dagger(\vec{k}) \mapsto \tilde{b}_s^\dagger(\vec{k}) = {b'_s}^\dagger(\vec{k})\eta \tag{44}$$

for all s and $\vec{k}$. This would change the anticommutation relations eq. (37) into commutation relations

$$[\hat{c}, \tilde{c}] = 0\,, \tag{45}$$

where $\tilde{c}$ is any operator of type $\tilde{a}_s(\vec{k})$, $\tilde{a}_s^\dagger(\vec{k})$, $\tilde{b}_s(\vec{k})$, or $\tilde{b}_s^\dagger(\vec{k})$.

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