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# The Einstein-Maximally Coupled Massive Scalar Field System on a Bianchi Type-I Cosmological Model

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**Abstract**

Bianchi type-I cosmological model with a maximally coupled real massive scalar field is investigated. We obtain, using an ad-hoc mathematical relation and a standard theory on first order differential equations, global solutions to the Einstein's field equations both at early and late cosmic times. The physical and geometrical behaviors of the cosmological model are also discussed.

**Mathematics Subject Classifications:** 83Cxx

**Keywords:** Bianchi type-I model; Maximally coupled massive scalar field; Differential equation; Global solution; Inflation and expansion

# 1 Introduction

Our actual Universe is described by all recent observations to be in an accelerated expansion. This Universe is also geometrically described by the Freedman-Lemaitre-Robertson-Walker(FLRW) models where the expansion is the same for any direction. In this paper, we will be interested in a geometry described by the Bianchi type-I cosmological anisotropic models, for which the expansion of Universe is not the same according to the direction of observation. Concerning the matter content, we consider only the presence of scalar fields in the Universe. A massive scalar field$\phi$ whose mass is denoted by $m$; and whose contribution to the energy-momentum tensor is defined by the standard tensor $\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} + \frac{1}{2}m^2\phi^2 g_{\mu\nu}$, on the one hand; and an additional coupling of the scalar field to the curvature, whose coupling term is $R\phi^2$ and whose contribution to the energy-momentum tensor is $2\left[g_{\mu\nu}\Box\phi^2 - (\phi^2)_{;\mu\nu}\right] + 2\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\phi^2$. The scalar field is then said to be maximally coupled. A scalar field is in fact a function which associates at each spatial-temporal point a number. A good example can be the temperature of a room : for each point of a room, one can associate a quantity T defining its temperature. The physical interest of scalar fields and Bianchi models is based on the fact that the study of the related Einstein-scalar field equations can help us to give a better description of the physical and geometrical properties of those models in order to confirm or discard the actual observations concerning the expansion and the acceleration of Universe.

We will see in our investigations that, the cosmological constant is not alone the most serious and natural candidate for explaining the cosmic acceleration of Universe$[5-6]$. A wide range of observations has suggested that the Universe possesses a non-zero cosmological constant $\Lambda$, which is considered as a measure of the energy-density of the vacuum[6].

But we reach the conclusion that, in the absence of the cosmological constant, all the scale factors which are important for cosmological observations are still monotonic functions of cosmic time. This shows that the cosmological evolution of our models is in expansion or in inflation.

Instead of using only the method followed by Bali and Jain[2], Sharif and Zubair [3], Adhav et al.[4] who used the physical condition that the shear scalar is proportional to the expansion scalar, we follow Hajj-Boutros [1]by using an ad-hoc reasonable mathematical relation. The advantage is that this method leads us almost to the same relation between the metric potentials $A$ and $B$ subject to the Einstein equations, obtained by $[2-3-4]$ and also gives us physically realistic solutions of the field variables in term of the scalar field.

Several authors have studied for a long time Bianchi models and various scalar-tensor theories$[7-16]$. *The most relevant thing we bring in this work is that we have additionally coupled the scalar field to the curvature and we have* *obtained, using a standard theory on ordinary differential equations, a global in time solution of our model both at early and late times. We have also seen that the model can reach a big-bang period since, in our investigations, we have proved that the model passes through inflation epoch or expansion one.*

The paper organizes as follows :

In section 2, we present the metric and the field equations.

In section 3, we investigate solutions to the field equations.

In section 4, we discuss about the behavior of the model obtained.

## 2 The metric and field equations

We consider the spatially homogeneous and anisotropic Bianchi type-I metric in the form:

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 \left(dy^2 + dz^2\right) \tag{1}$$

where $A, B$ are functions of cosmic time $t$ alone.

The energy-momentum tensor for a maximally coupled and massive real scalar field in vacuum is of form:

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} + \frac{1}{2}m^2\phi^2 g_{\mu\nu} + 2\left[g_{\mu\nu}\Box\phi^2 - \left(\phi^2\right)_{;\mu\nu}\right] + 2\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\phi^2 \tag{2}$$

in which $\phi$ stands for the real massive scalar field whose mass is denoted by $m > 0$.

The corresponding Lagrangian writes :

$$L = \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}m^2\phi^2 + R\phi^2 \tag{3}$$

where $R\phi^2$ is the coupling term and induces the following contribution to the energy-momentum tensor:

$$\triangle T_{\mu\nu} = 2\left[g_{\mu\nu}\Box\phi^2 - \left(\phi^2\right)_{;\mu\nu}\right] + 2\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\phi^2. \tag{4}$$

Notice that $\phi$ is always subject to the equation of motion $\frac{\partial L}{\partial \phi} = 0$ which reads:

$$\Box\phi + m\phi\dot{\phi} - 2\phi R = 0, \tag{5}$$

where, here and in what follows, a dot denotes the derivative with respect to $t$.

The Einstein-Maximally coupled massive real scalar field system of equations ( with $8\pi G = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{6}$$

together with equation (2) for the line-element (1) lead to:

$$\left(1-2\phi^2\right)\left(2\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2\right) = \frac{\dot{\phi}^2 - m^2\phi^2}{2} + 4\phi\dot{\phi}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \tag{7}$$

$$\left(1-2\phi^2\right)\left(2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2\right) = -\frac{\dot{\phi}^2 + m^2\phi^2}{2} + 4\left(\dot{\phi}^2 + \phi\ddot{\phi} + 2\frac{\dot{B}}{B}\phi\dot{\phi}\right) \tag{8}$$

$$\left(1-2\phi^2\right)\left(\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB}\right) = -\frac{\dot{\phi}^2 + m^2\phi^2}{2} + 4\left(\dot{\phi}^2 + \phi\ddot{\phi} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\phi\dot{\phi}\right). \tag{9}$$

The equation (7) is the Hamiltonian constraint, and it is well known that this equation is automatically satisfied if the initial data $A_0 = A\left(0\right), B_0 = B\left(0\right), \dot{A}_0 = \dot{A}\left(0\right), \dot{B}_0 = \dot{B}\left(0\right), \phi_0 = \phi\left(0\right), \dot{\phi}_0 = \dot{\phi}\left(0\right)$ verify the equation:

$$\left(1-2\phi_0^2\right)\left(2\frac{\dot{A}_0\dot{B}_0}{A_0B_0} + \left(\frac{\dot{B}_0}{B_0}\right)^2\right) = \frac{\dot{\phi_0}^2 - m^2\phi_0^2}{2} + 4\phi_0\dot{\phi}_0\left(\frac{\dot{A}_0}{A_0} + 2\frac{\dot{B}_0}{B_0}\right). \tag{10}$$

One requires in what follow that the initial data verify equation $(10)$.

Now we define some parameters for the Bianchi type-I model which show useful in cosmological observations.

The average scale factor $a$ and the spatial volume of Universe $V$ are defined as:

$$V = a^3 = AB^2. \tag{11}$$

The expansion scalar $\theta$ and shear scalar $\sigma^2$ are given by the following expressions:

$$\theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}, \tag{12}$$

$$\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2. \tag{13}$$

We also define the generalized mean Hubble parameter $H$ by:

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{14}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = H_3 = \frac{\dot{B}}{B}$ are the directional Hubble's parameters in the directions of $x, y$ and $z$ respectively.

The most physical quantity of observational interest in cosmology is the deceleration parameter $q$ defined by:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{d}{dt}\left(\frac{1}{H}\right). \tag{15}$$

It is important to stress here that the sign of $q$ indicates whether the model is in inflation or not. The positive value indicates the decelerating model whereas the negative value shows acceleration.

# 3 Solutions to the field equations

The system of equations $(7)-(8)-(9)$ is a system of three non-linear differential equations in three unknowns $A, B$ and $\phi$.

Since the equation (7) is a property of solutions of the other equations $(8), (9)$, and is always verified by any solutions $A, B, \phi$ of $(8)-(9)$, we are going to use following [1], and ad-hoc mathematical relation to obtain physically realistic solutions of the field variables.

From (8) and (9), we obtain:

$$\left(1 - 2\phi^2\right)\left(\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \left(\frac{\dot{B}}{B}\right)^2 - \frac{\dot{A}\dot{B}}{AB}\right) = 4\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\phi\dot{\phi}. \tag{16}$$

In fact, equation (16) is automatically verified if we insert the ad-hoc mathematical relation:

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = 0. \tag{17}$$

Equation (17) integrates to give:

$$B = c^2 A, \tag{18}$$

where $c^2 \neq 1$ is a constant of integration, because we are not dealing with the FLRW model.

Equation (7), which is always satisfied, reduces then to the following:

$$3\left(1-2\phi^2\right)\left(\frac{\dot{A}}{A}\right)^2-12\phi\dot{\phi}\frac{\dot{A}}{A}-\frac{\dot{\phi}^2-m^2\phi^2}{2}=0. \tag{19}$$

This means that equation (19) which unknown is this time $\frac{\dot{A}}{A}$, always has one or two different solutions.

Its possesses a unique solution:

$$\frac{\dot{A}}{A}=\frac{2\phi\dot{\phi}}{1-2\phi^2} \tag{20}$$

if and only if:

$$22\dot{\phi}^2\phi^2+\dot{\phi}^2+2m^2\phi^4-m^2\phi^2=0. \tag{21}$$

Equation (21) is a first order differential equation that can be written in the form:

$$\dot{\phi}=f\left(t,\phi\right), \tag{22}$$

in which

$$f^2\left(t,\phi\right)=\frac{m^2\left(\phi^2-2\phi^4\right)}{1+22\phi^2}. \tag{23}$$

The relation (23) shows that $f$ is a continuous function of $t$ and locally Lipschitzian in $\phi$.

By applying the standard theory on first order differential equations, there exists a maximal interval $[-T,T]$, $T>0$, such that the equation (22) has a *unique local solution* $\phi$ defined on $[-T,T]$ and verifying:

$$\phi\left(0\right)=\phi_0. \tag{24}$$

Now, it is easily seen by (23) that any solution $\phi$ of (22) verifies:

$$-\frac{\sqrt{2}}{2}\leq\phi\leq\frac{\sqrt{2}}{2}. \tag{25}$$

This proves that $\phi$ is *uniformly bounded* on $]-T,T[$ and consequently $\phi$ is **global** on $]-\infty,+\infty[$.

By equation $(19)$, together with relation $(18)$, $A$ and $B$ are computed to give:

$$A=\frac{K^2}{\sqrt{1-2\phi^2}},\ B=\frac{K^2c^2}{\sqrt{1-2\phi^2}}, \tag{26}$$

where $K^2$ is a constant of integration.

This proves that, because $\phi$is global, solutions $A$ and $B$ of the Einstein equations $(8) - (9)$ are also **global**.

The model (1) then writes:

$$ds^2 = -dt^2 + \frac{K^4}{1 - 2\phi^2} dx^2 + \frac{K^4 c^4}{1 - 2\phi^2} \left(dy^2 + dz^2\right). \tag{27}$$

# 4 Discussion

We observe first of all that, by equation $(21)$ :

$$\dot{\phi} = \pm\sqrt{\frac{m^2\left(\phi^2 - 2\phi^4\right)}{1 + 22\phi^2}}. \tag{28}$$

This proves that $\phi$ is always a monotonic function of time $t$.

Now by relation $(20)$, if $\phi$ has a constant sign ( this means $\phi \in \left[-\frac{\sqrt{2}}{2}, 0\right]$or $\phi \in \left[0, \frac{\sqrt{2}}{2}\right]$), then $\frac{\dot{A}}{A}$ is also always monotonic in time $t$.

The relations $(11), (12), (13), (14), (17)$ together with equation $(18)$ show that:

$$V = c^4 A^3,\, \theta = 3\frac{\dot{A}}{A} = 3H,\, \sigma = 0. \tag{29}$$

By relation $(25)$, $\phi$ is bounded, so $V$ is not zero both for finite or large values of time $t$, and so the model (27) has no singularity at finite or large (early and late) times.

If $\phi$ has a constant sign, then by (29) all scale factors are monotonically increasing or decreasing functions of time $t$, so the model always inflates or expands.

$\frac{\sigma}{\theta} = 0$, so the model does not approach isotropy for large time.

We can not say anything about acceleration or deceleration of the expansion or the inflation, since we have not investigated about the sign of the parameter $q$.

# 5 Conclusion

In the present paper, we have presented a global solution for the Einstein's field equations for a spatially homogeneous and anisotropic Bianchi type-I space time in the case of a maximally coupled and massive real scalar field. We have seen that, by making adequate considerations on the scalar field, the model expands, inflates and is shearing. The cosmological model has no finite or large time singularity and does not reach isotropy . Our investigations also show that the model passes probably through a big-bang epoch.

The present model may be useful to describe the early and late stages of the evolution of our physical Universe, since, it can explain the vacuum gravity state, confirm the fact that scalar fields have been present in our earlier Universe. It also makes possible to understand the mechanism which led the early Universe to the large scale structure and predicts the fate of the whole universe.

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