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The No-boundary Proposal via the Five-dimensional Friedmann-Lemaître-Robertson-Walker Model

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Abstract

Hawking's proposal that the Universe has no temporal boundary and hence no beginning depends on the notion of imaginary time and is usually referred to as the *no-boundary proposal*. This paper discusses a simple alternative approach by means of the five-dimensional Friedmann-Lemaître-Robertson-Walker model.

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1 Introduction

It is well known that Stephen Hawking introduced the no-boundary proposal using the concept of imaginary time, making up what is called Euclidean spacetime [1]. While ordinary time would still have a big-bang singularity, imaginary time avoids this singularity, implying that the Universe has no temporal boundary and hence no beginning. By eliminating the singularity, the Universe becomes self-contained in the sense that one would not have to appeal to something outside the Universe to determine how the Universe began.

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Given that imaginary time is orthogonal to ordinary time, one could view imaginary time as an extra dimension. The existence of an extra dimension suggests a different starting point, the five-dimensional Friedmann-Lemaître-Robertson-Walker model (FLRW) [2]. This model produces a simple alternative version of the no-boundary proposal.

2 The FLRW model

Let us recall the FLRW model in the usual four dimensions [3]:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1}$$

where $a^2(t)$ is a scale factor. Here $K = 1/R^2$, where

$$R = \sqrt{x^2 + y^2 + z^2 + w^2}; \tag{2}$$

also, $K > 0$, $K = 0$, and $K < 0$ (imaginary R) correspond to a closed, flat, and open universe, respectively. For $K > 0$, the substitution $r = \frac{1}{\sqrt{K}}\sin\psi$ yields

$$ds^2 = -dt^2 + a^2(t)R^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{3}$$

The spatial part of the metric is a 3-sphere, having neither a center nor an edge. Here we need to recall that a 3-sphere can be defined as a three-dimensional boundary of a fictitious four-dimensional ball of radius R, as defined in Eq. (2). In the discussion below we will assume a unit sphere, i.e., $R = 1$.

Returning to line element (3), observe that the singularity in line element (1) has been removed, showing that we are dealing with a coordinate singularity, not a physical one. The case $K < 0$ does not lead to a singularity. Here the substitution $r = \frac{1}{\sqrt{|K|}}\sinh\psi$ leads to the analogous line element in hyperbolic coordinates (with $R = 1$):

$$ds^2 = -dt^2 + a^2(t)[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{4}$$

Turning now to the five-dimensional spatially homogeneous and isotropic FLRW metric, we have, according to Ref. [2],

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-Kr^2} + r^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\right\}. \tag{5}$$

The above substitutions for r now lead to the respective line elements

$$ds^2 = -dt^2 + a^2(t)\left\{d\chi^2 + \sin^2\chi[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\right\}. \tag{6}$$

and

$$ds^2 = -dt^2 + a^2(t)\left\{d\chi^2 + \sinh^2\chi[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\right\}. \quad (7)$$

The respective spatial parts can be viewed as four-dimensional boundaries of a five-dimensional unit sphere and a five-dimensional unit hyperboloid.

Ref. [2] makes the usual assumption that any extra spatial dimension has been compacted to small size in the course of the evolution of the Universe. Since the extra dimension does not participate in the expansion, more realistic line elements are obtained by leaving $a^2(t)$ in the original position:

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi\left\{a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\right\} \quad (8)$$

or

$$ds^2 = -dt^2 + d\chi^2 + \sinh^2\chi\left\{a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\right\}. \quad (9)$$

3 The no-boundary proposal

Stephen Hawking's proposal that the Universe had no beginning [1] depends on the notion of imaginary time. If one thinks of ordinary time as a real axis pointing to the future in one direction and the past in the other, then the imaginary-time axis is perpendicular to the real-time axis. The main idea in this note is to show that the existence of an extra spatial dimension can also lead to the no-boundary proposal.

Since our approach is heavily dependent on the time-component, we first recall that in geometrized units ($G = c = 1$) the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

describes a black hole having an event horizon at $r = 2M$. Ordinarily, $r > 2M$. Inside the event horizon we have $r < 2M$, so that the first two terms undergo sign changes. So t becomes spacelike and r timelike, leading to the traditional argument that motion in the r-direction cannot be reversed, so that escape from a black hole is impossible.

To show that the time component can also become spacelike in the present study, let us recall the Lorentzian metric

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \quad (\text{signature: } - + + +)$$

or

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (\text{signature: } + - - -)$$

temporarily reintroducing c. (The choice of signature is merely a matter of convenience.) In spherical coordinates,

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

or

$$ds^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

In the Minkowski system (ict, r, θ, ϕ), the former can be written

$$ds^2 = (icdt)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

thereby retaining the Euclidian form. So the metric (3) becomes (since $R = 1$)

$$ds^2 = (icdt)^2 + a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{10}$$

The general theory of relativity is based on a four-dimensional pseudo-Riemannian geometry. With an extra spatial dimension, the natural analogue of a 3-sphere is a 4-sphere or a 4-hyperboloid. To obtain this analogue, we would have to include (with $c = 1$ again) both $(idt)^2$ and $\sin^2(it)$ or both $(idt)^2$ and $\sinh^2(it)$ in order to retain the required form of the line element.

Since the extra spatial dimension is not affected by the expansion, the scale factor $a^2(t)$ remains in its original position. However, the forms of the resulting metrics require an extra term, here denoted by $d\eta^2$:

$$ds^2 = d\eta^2 + (idt)^2 + \sin^2(it)\{a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\}. \tag{11}$$

and

$$ds^2 = d\eta^2 + (idt)^2 + \sinh^2(it)\{a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]\}. \tag{12}$$

(Using the identities $\sin(it) = i\sinh t$ and $\sinh(it) = i\sin t$, these line elements could also be written with signatures $+----$ or $-++++$.) Eqs. (11) and (12) have the forms of a 4-sphere and a 4-hyperboloid, respectively. The respective spatial parts therefore represent a four-dimensional boundary of a five-dimensional ball and hyperboloid. Observe that t has become spacelike.

4 Discussion

The FLRW expanding closed Universe is a 3-sphere. Like the surface of an expanding balloon, there is no edge and every point has the appearance of a center that all the other points on the surface recede from. So the best way to describe a 3-sphere is a three-dimensional boundary of a four-dimensional ball. There is no edge and every point looks like the center of the Universe. The open case leads to a 3-hyperboloid.

In this note we considered the Euclidean form of the FLRW model with $c = 1$:

$$ds^2 = (idt)^2 + a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

or

$$ds^2 = (idt)^2 + a^2(t)[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

Since the absence of a spatial edge is consistent with observation, the extra spatial dimension suggests an extension of the FLRW model that results in the line elements (11) and (12). These ideas lead to the following possible interpretations:

4.1 Interpretation 1

The respective spatial parts in (11) and (12) represent a four-dimensional boundary of a five-dimensional sphere or hyperboloid. Being a boundary, there is, once again, no edge. The difference is that the boundary includes the t in the *original* Lorentzian spacetime, which implies that the Universe cannot have an edge, either spatial or temporal. The absence of a temporal edge is consistent with Hawking's no-boundary proposal.

The required form of the metric forced the introduction of a new component in the five-dimensional model, Eqs. (11) and (12). The original time t is, as in Hawking's theory, an illusion, making the five-dimensional time η the "real" time; moreover, the new time is orthogonal to the original time.

4.2 Interpretation 2

Alternatively, one could view the five-dimensional space as a mathematical abstraction whose only purpose is to define the four-dimensional boundary, just as a four-dimensional fictitious sphere is used to define the boundary of a 3-sphere. With this interpretation, there is no physical second time component, leaving only ordinary time, but there is still no temporal edge and hence no beginning.

From the standpoint of the cosmological principle, Interpretation 1 is probably preferred. Recall that according to this principle, there is no such thing as a special *place*: everything looks essentially the same in every direction for every observer. Yet strictly speaking, everything should look the same at any *time*, as well. Returning to t, thanks to the Big Bang, this aspect of the cosmological principle is lost: the Universe would look different for any two observers who are widely separated in time, i.e., at respective times t_1 and t_2, where $t_1 \ll t_2$. No such effect exists for the new time η, however, so that η reestablishes the "perfect cosmological principle."

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