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Schrödinger-like Interpretation of Scalar Field Equation in RW Space-time

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Abstract

The scalar field equation in Robertson-Walker space-time is put, by elementary transformation, into the the form of covariant eigenvalue probem of ordinary QM. The spectrum of the corresponding Schrödinger operator is determined according to the value of the curvature parameter $a = 0, \pm 1$. There results that the energy E is positive non degenerate in all cases. For $a = 0$, there results $E > 0$. Instead in the curved cases, E cannot be arbitrarily small: one has $E > 1/2$ for $a = -1$ and $E = E_j = (j+2)^2 - 1,\ j = 0, 1, 2, ...$ If of physical meaning the results seem to be better interpreted in case of massless particle and would discriminate the curvature in the standard cosmological model formulation.

Keywords: Scalar Field in RW; Schrödinger Quantization; Solutions

1 Introduction

The scalar field equation in Robertson-Walker (RW) space-time has been widely considered being a basic quantum wave equation in curved space-time. Its integration is generally performed by variable separation (see e. g., [2, 9]). Also the quantization of the scalar field equation in curved space-time has been widely studied (see e.g., [2, 5]). As far as the author knows, the reduction of the scalar field equation to a one dimensional Schrödinger like eigenvalue problem has not yet been considered. Such procedure was developed for the Dirac equation in Kerr geometry [3] and, successively, also in RW geometry [4, 10].

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In case of the scalar field the study may be of interest because it represents an alternative quantization to the canonical field quantization in curved space-time. To that end, the scalar field equation being separable, one can confine to work on the radial equation. By elementary changes of the variable and of the wave function, the separated radial equation is here put into the form of a Schrödinger like eigenvalue equation. It is such equation that is interpreted in the ordinary quantum way. The solution is obtained for each value of the curvature parameter $a = 0, \pm 1$. For $a = 0, -1$ one has only scattering state solutions. The corresponding energy spectrum is non degenerate and consists of the values $k^2 > 0$ if $a = 0$ and $k^2 > 1$ if $a = -1$. For $a = 1$ the discrete non degenerate spectrum and the corresponding proper states are determined. The discrete energy levels are very similar to the asymptotic ones of the mass less neutrino for zero angular momentum [4].

2 Assumptions and preliminary results

The equation for the scalar field $\phi(x)$ non minimally coupled to gravity in a general curved space time reads

$$\nabla_\alpha \nabla^\alpha \phi(x) + [m_0^2 + \zeta \bar{R}(x)]\phi(x) = 0 \tag{1}$$

where ∇_α is the covariant derivative relative to the metric tensor $g_{\mu\nu}$, $\bar{R}(x) = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, ζ a numerical real factor and m_0 the mass of the field. In case of the Robertson-Walker (RW) metric whose $g_{\mu\nu}$ is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - R^2(t)\Big[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2\Big], \quad a = 0, \pm 1 \tag{2}$$

the equation (1) can be separated by variable separation. Indeed (e.g. [9]) one is left with

$$\phi = T(t)S(r)Y_{lm}(\theta, \varphi), \quad l = 0, 1, 2.., \; m = -l, -l+1, ..0, 1, ..l \tag{3}$$

$$\ddot{T}R^2 + 3R\dot{R}\dot{T} + [k^2 + m_0^2 R^2 + 6\zeta(\ddot{R}R + \dot{R}^2 + a)]T = 0 \tag{4}$$

$$(1 - ar^2)S'' + \Big(\frac{2}{r} - 3ar\Big)S' + \Big(k^2 - \frac{\lambda}{r^2}\Big)S = 0 \tag{5}$$

Y_{lm} the usual spherical harmonics, k, λ the separation constants for the time and angular equations. There results $\lambda = l(l+1)$, $l = 0, 1, 2, ...$ As to the solution of (4), some developments can be given when $R(t)$ describes the time evolution of the standard cosmological model [9]. The equation (5) has been solved exactly for $a = 0, \pm 1$ (see references in [9]). On can then select a complete set of solutions ϕ_α of (1) orthogonal in a suitable scalar product that provides the basis for covariant quantization of the scalar field in analogy to the Minkowski space time case (e. g., [9]).

Instead here the equation (5) will be transformed into a Schrödinger like eigenvalue problem by elementarily changing the independent variable and radial wave function. Define then

$$s = \int \frac{dr}{\sqrt{1-ar^2}} \tag{6}$$

If $a = 0$, $r = s$. By setting $S(r) = F(r)/r$, from (5) one has

$$-\frac{d^2F}{dr^2} + \frac{4 + l(l+1)}{r^2}F = k^2F \tag{7}$$

If $a = -1$, $s = \sinh^{-1} r$. If $S(s) = F(s)/\sinh s$, (5) gives

$$-\frac{d^2F}{ds^2} + \Big(1 + \frac{l(l+1)}{\sinh^2 s}\Big)F = k^2F \tag{8}$$

If $a = 1$, $s = \sin^{-1} r$. With $S(s) = F(s)/\sin s$, (5) implies

$$-\frac{d^2F}{ds^2} + \Big(\frac{l(l+1)}{\sin^2 s} - 1\Big)F = k^2F \tag{9}$$

The equations (7), (8), (9) have the form of a Schrödinger like eigenvalue problem. They are interpreted as the quantum scalar field equation in RW space-time. The spectra of the values of k^2 are accordingly determined by requiring the eigenstates (or their eigenpackets) to belong to $L^2((0,\infty), ds)$ for $a = 0, -1$ or to $L^2((0,\pi/2), ds)$ for $a = 1$. On the base of qualitative results on one dimensional Schrödinger operators (e. g., [6]), acceptable solutions must vanish in $s = 0$ for $a = 0, \pm 1$ and also in $s = \pi/2$ for $a = 1$. The expression $E = k^2/2$ will be interpreted as the energy of the system.

3 Solution for the flat space time case

The equation (7) can be reported to a confluent hypergeometric equation [1] through the steps:

$$F = r^\alpha e^{ikr} Z(r), \qquad \alpha = (1 \pm \sqrt{(2l+1)^2})/2 \tag{10}$$

$$\xi = -2ikr \tag{11}$$

$$\xi Z'' + (2\alpha - \xi)Z' - \alpha Z = 0 \tag{12}$$

There are two independent solutions of (7):

$$F_{1k} = r^\alpha e^{ikr} \phi(\alpha; 2\alpha; -2ikr) \tag{13}$$

$$F_{2k} = r^{1-\alpha} e^{ikr} \phi(1-\alpha; 2-2\alpha; -2ikr) \tag{14}$$

If $\alpha > 0$ in (10) then F_{2k} is not acceptable because it does not vanish in $r = 0$. On the other hand [1]

$$F_{1k} \quad \overset{r\to\infty}{\longrightarrow} \quad \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\Big\{e^{-ikr}(-2ik)^{-\alpha} + e^{ikr}(2ik)^{\alpha}\Big\} \tag{15}$$

If k is real F_{1k} is not of class L^2 in $r = \infty$ but its eigenpackets ΔF_{1k} do are. If $k = i\chi$ nor F_{1k} nor its eigenpackets are of L^2-class. (If $\alpha < 0$ the role of F_{1k} and F_{12k} are reversed.) Therefore (7) has a continuum non degenerate spectrum of eigenvalues $k^2 > 0$. The corresponding solutions are improper (scattering) states.

4 Solution for the open space time case

The equation (8) can now be reported to a hypergeometric equation through the steps:

$$y = \cosh s \tag{16}$$
$$F = (1-y^2)^{\alpha}u(y), \qquad \alpha = (l+1)/2 \tag{17}$$
$$(y^2-1)u'' + (1+4\alpha y)u' + (2\alpha+\lambda+k^2-1)u = 0 \tag{18}$$

Finally with $1 - y = 2x$, (18) becomes

$$x(1-x)u'' + [c-(a+b+1)x]u' - ab\,u = 0 \tag{19}$$
$$a = l+1 \pm \sqrt{1-k^2}, \quad b = l+1 \mp \sqrt{1-k^2}, \quad c = l+3/2 \tag{20}$$

Independent solutions of (8) are

$$F_{1k} = (1-y^2)^{\frac{l+1}{2}} F(a, b; c; (1-y)/2) \tag{21}$$
$$F_{2k} = (1+y)^{\frac{l+1}{2}}(1-y)^{-\frac{l}{2}} F(1-c+a, 1-c+b; 2-c; (1-y)/2) \tag{22}$$

Since F must vanish for $s = 0$, F_{2k} is not acceptable. On the other hand

$$F_{1k} \quad \overset{s\to\infty}{\longrightarrow} \quad \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\Big(\frac{e^s}{4}\Big)^{\mp\sqrt{1-k^2}} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\Big(\frac{e^s}{4}\Big)^{\pm\sqrt{1-k^2}} \tag{23}$$

where only the upper or only the lower signs have to be considered toghether. Therefore if $k^2 < 1$ nor F_{1k} nor its eigenpackets ΔF_{1k} are of class L^2. If instead $k^2 > 1$ then F_{1k} is an oscillating function for $s \to \infty$. It is not of L^2-class but its eigenpackets do are. The spectrum of the eigenvalues of (8) is then the continuous non degenerate spectrum of values $k^2 > 1$.

5 Solution for the closed space time case

By setting $y = \cos s$ in (9) one can proceed as for $a = -1$ thus obtaining the equations (19)-(22) with the only difference of the definition of the $y(s)$. In the present case acceptable solutions must vanish for $s = 0, \pi/2$ $(y = 1, 0)$. Therefore, from (22), F_{2k} is not acceptable. Instead F_{1k} vanishes in $y = 1$ and in order to be acceptable it must be $F_{1k}(y = 0) = 0$. Since from (20) $c = (a + b + 1)/2$ one has to have [1]

$$F(a, b; \tfrac{a+b+1}{2}; \tfrac{1}{2}) = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{a}{2} + \frac{b}{2})}{\Gamma(\frac{1}{2} + \frac{a}{2})\Gamma(\frac{1}{2} + \frac{b}{2})} = 0 \tag{24}$$

that can be satisfied for $(a + 1)/2 = -n,\ n = 0, 1, 2...$ or $k^2 \equiv k_n^2 = (2n + l + 2)^2 - 1$. Such set of eigenvalues can be recast into the form

$$k_n^2 = (j + 2)^2 - 1, \quad j = 0, , 1, 2, ... \tag{25}$$

The corresponding proper eigenstates $F_{1n}(s)$ are then [1]

$$(\sin s)^{l+1} F\Big(-n, 2l + 2 + n; l + \frac{3}{2}; \frac{1 - \cos s}{2}\Big) = \frac{n!(\sin s)^{l+1}}{(2l + 2)_n} C_\alpha^{l+1)}(\cos s) \tag{26}$$

and results already orthogonal and complete in $L^2((0, \pi/2), ds)$ on account of the orthogonality relations of the Gegenbauer polynomials [1]). Therefore the spectrum is, as expected, proper, positive non degenerate.

6 Concluding Remarks

The scalar field equation in RW space-time has been elementary reduced to a Schrödinger like eigenvalue problem. The main consequence is the nature of the spectrum of the associated Schrödinger operator that results non degenerate in every case of the curvature parameter. In the flat space time case the eigenvalues k^2 are such that $k^2 > 0$. For $a = -1$ there results $k^2 > 1$ while for $a = 1$ one has $k^2 = k_{nl}^2 = (2n + l + 2)^2 - 1$ (or, equivalently, (25)). Therefore in the non flat space-time cases the energy cannot be arbitrarily small. The expression of E_n for $a = 1$ is very similar to the $l = 0$ asymptotic expression of the energy spectrum of the massless neutrino in RW space time [4].

In every case of the curvature parameter, the energy spectrum is independent of the mass of the particle, whose role is confined in the time evolution equation (4). Even if problematic, such aspect could be ascribed to an a priori non impossible cosmological effect. In case of massless fields, the interpretation of the above scheme seems to be more consistent. In particular, if of physical meaning, it could represent a further property of Goldstone Bosons [8].

Finally, if there where an experimental evidence that massless spin zero particle of cosmological origin cannot approximate the zero value of the energy, this would support the idea that the standard cosmological model is flat.

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