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LTB Model with Pressure and Particle Creation Term

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Abstract

A Lemaître-Tolman-Bondi (LTB) cosmological model with non trivial pressure and particle creation assumption is considered. A priori the presence of pressure extends the LTB model while particle creation assumption seems to specialize it. The object is to integrate the scheme. This is achieved in general if $\rho(r,t) = -p(r,t)$ finding non factorized solutions. If $p = p(t)$ also ρ depends only on t , the physical radius is of the form $y(r,t) = T(t)F(r)$, the scheme is separable and one is formally reduced to a Robertson-Walker metric with generalized radial coordinate. The interest lies then on the time dependence of the model in case $p(t) = w\rho(t)$. The resulting coupled time equations in $\rho(t)$ and $T(t)$ are disentangled and the explicit solution is determined.

Keywords: LTB cosmology; Particle creation term; Solution of the equation

1 Introduction

Particle production by universe expansion is a well known quantum effect that is consequence of field quantization in curved space-time [5]. Recently the number $u_{sk}(t)$ of particles of spin s, momentum k created by universe expansion per unit time at time t has been calculated in Robertson-Walker (RW) space-time, the result being $u_{sk}(t) = \pm 6\,\dot{R}(t)/R(t)$, $R(t)$ the cosmic scale factor. The result holds for spin 0, 1/2, 1 fields (and it is expected to

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hold for higher spin) ([13] and references therein). Similar discussion have been developed in Lemâitre-Tolmann-Bondi (LTB) metric where the same result has been obtained for $s = 1$ and where it is expected to hold for $s = 0, 1/2$ [9]. In spite of some limitations (e. g., the number of created particles per unit time in RW is divergent and in the LTB model it has been proved only for the lower spin values) the result has been taken into account in the formulation of cosmological model. In Standard Cosmology as well as in LTB Cosmology the contribution of particle creation has been achieved in essentially two ways. The first one consists in performing the substitution $\rho \to \rho + \alpha(t)\dot{R}/R$ in RW and $\rho \to \rho + \alpha(t)\dot{y}/y$ in LTB cosmology, $y = y(r,t)$ then physical radius. The second one consists in adding to the Einstein field equation a cosmological term of the form $\alpha\dot{R}/R$ in RW and $\alpha\dot{y}/y$ in LTB cosmology. In RW cosmology the second formulation seems to better fit the experimental values of the cosmological parameters [13] even if a complete solution and discussion of both formulation has not yet been obtained (see the comments [11, 12]).

In the conventional LTB cosmology the substitution $\rho \to \rho + \alpha\dot{y}/y$ leads to calculations that become rapidly difficult and involved. A study of special cases of such model has been discussed in [10]. Instead the formulation of back reaction to particle creation by a cosmological term $\propto \dot{y}/y$ drastically reduces the generality of the scheme to a Robertson-Walker like metric with generalized radial function [10]. It remains of interest the surviving separated time equations that was solved only in special cases [14]. In the mentioned papers on the LTB model, numerical results have not been explicitly reached to be compared with the experimental data. The model seems however of interest. Recently the no homogeneity of the LTB model has been considered an alternative to the introduction of dark energy and dark matter as done in Standard Cosmology (e. g., [1, 6]). It seems therefore of interest to go further in the study of the model.

The object of the present paper is of studying an extended LTB model that maintains the formulation of back reaction to particle creation in the second way just mentioned, but that avoids the implied specialization of the scheme. This is done by simply assuming non zero pressure of dust matter of the form $p = p(r,t)$. There results a non trivial generalization, mainly for mathematical aspects, of the scheme studied in [14]. The model is again reduced to a generalized Kepler-like equation with variable mass and velocity dependent term with a very involved y dependence. The case $p(r,t) = -\rho(r,t)$ is integrated exactly. Due to analytical complexity this is the only case where we find non factorized y- solutions. The study is then specialized to the assumption $p = p(t)$ that again, unfortunately, confines the scheme to the case of absence of pressure. Indeed ρ depends only on the time, the physical radius is factorized $y = T(t)F(r)$ and the entire model can be integrated by variable separation. The choice $p(t) = w\rho(t)$ is then considered. The solution is reduced to the solution of a pair of non linear coupled differential equations in ρ, T. The equations are disentangled and one is finally left with the study of a nonlinear second order differential equation in T that is integrated.

2 Cosmological model

The object is to describe a spherically symmetric Universe filled with freely falling dust-like matter of density ρ and pressure p where particle creation is admitted. The mathematical framework is that of spherically symmetric comoving coordinate systems of line element [8]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^{\gamma(r,t)}dr^2 - y^2(r,t)[d\theta^2 + \sin^2\theta\, d\varphi^2] \qquad (1)$$

By symmetry one has a priori $\rho = \rho(r,t),\ p = p(r,t)$. The dynamics of the Universe is governed by the Einstein field equation ($\dot{y} = \partial y/\partial t$)

$$R_{\mu\nu} - \frac{1}{2}R^{\ \beta}_{\beta}\, g_{\mu\nu} = k\, T_{\mu\nu} + k\lambda\frac{\dot{y}}{y}\, g_{\mu\nu}, \qquad |k| = 8\pi G \qquad (2)$$

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - p\, g^{\mu\nu}, \qquad U^0 = 1,\ U^j = 0,\ j = r, \theta, \varphi \qquad (3)$$

$$R_{\mu\nu} = k(\rho + p)\, U_\mu U_\nu - k\, g_{\mu\nu}\, [\frac{\rho - p}{2} + \lambda\frac{\dot{y}}{y}] \qquad (4)$$

As discussed in the Introduction $k(\dot{y}/y)g_{\mu\nu}$ is interpreted as a back reaction term proportional to the "number" of particle created at time t. $T^{\mu\nu}$ is the usual energy momentum tensor of a perfect fluid of density ρ and pressure p in comoving coordinates [8]. The equation (4) is a readjustment of (2) after taking the trace. Making explicit (4) in the metric (1) gives:

$$\frac{2\ddot{\gamma} + \dot{\gamma}^2}{4} + 2\frac{\ddot{y}}{y} = \frac{k}{2}(\rho + 3p) - k\lambda\frac{\dot{y}}{y} \qquad (5)$$

$$\frac{2yy'' - y'\gamma' y}{y^2} - e^\gamma(\frac{\dot{\gamma}^2}{4} + \dot{\gamma}\frac{\dot{y}}{y} + \frac{\ddot{\gamma}}{2}) = k\, e^\gamma\Big(\frac{\rho - p}{2} + \lambda\frac{\dot{y}}{y}\Big) \qquad (6)$$

$$-1 + \frac{e^{-\gamma}}{2}[2yy'' + 2y'^2 - yy'\gamma'] - y\ddot{y} - \dot{y}^2 - \frac{y\dot{y}\dot{\gamma}}{2} = ky^2\Big(\frac{\rho - p}{2} + \lambda\frac{\dot{y}}{y}\Big) \qquad (7)$$

$$\exp\gamma(r,t) = y'^2/(1 + 2E(r)) \qquad (8)$$

$E(r)$ is an arbitrary integration function. The left terms of (5), (6), (7) are the expression of $R_{\alpha\alpha}$, $\alpha = t, r, \theta$ while (8) is the integral of $R_{rt} = 0$. Consistency of (2), $\nabla_\mu(T^\mu_\nu + \lambda g^\mu_\nu \dot{y}/y = 0$, gives ($y' = \partial y/\partial r$)

$$\partial_t(\rho y^2 y') = -\lambda y^2 y' \partial_t(\dot{y}/y) - p\partial_t(y^2 y'), \quad \nu = t \qquad (9)$$

$$\partial_r(p - \lambda\dot{y}/y) = 0, \qquad \nu = r \qquad (10)$$

The integration of (10) leads to

$$y(r,t) = \exp\Big\{f(t) + g(r) + \frac{1}{\lambda}\int^t p(r,t)dt\Big\} \tag{11}$$

where $f(t)$, $g(r)$ are arbitrary integration function.

3 Reduction of the equations

Using (8), equation (5) reduces to

$$(y^2\ddot{y})' \;=\; k\, y^2 y' \Big[\frac{\rho+3p}{2} - \lambda\frac{\dot{y}}{y}\Big] \tag{12}$$

By expressing $\gamma(r,t)$ as a function of y', E (by (8)) into (6), (7) and then combining the resulting equations one finally has

$$\frac{\dot{y}^2}{2} + k\frac{m(r,t)}{y} + \frac{k}{2}y^2\Big(\lambda\frac{\dot{y}}{y} - p\Big) = E \tag{13}$$

$$m(r,t) = \int^r dr\, y^2 y'\Big(\frac{\rho+3p}{2} - \lambda\frac{\dot{y}}{y}\Big) \tag{14}$$

(One can check that for $p = 0$ eqs. (13), (14) reduce to the situation of [13]). The equation (13), that is a Kepler-like equation with variable mass and velocity dependent term, is very involved since m depends on y through (14). Besides (13) the physical radius satisfies also (9) that can be rearranged in the form

$$\dot{\rho} + (\rho+p)\frac{\partial_t(y^2y')}{y^2y'} = -\lambda\,\partial_t\Big(\frac{\dot{y}}{y}\Big) \tag{15}$$

The equations can be solved once a state equation for ρ, p is given. At the present level of generality, that is by letting p to be function of both r and t, the solution seems a difficult task as far as the author is concerned. An exception is the following.

4 The case $\rho(r,t) = -p(r,t)$: exact solution

The scheme can be solved in the analog of the Vacuum Energy case of the Standard Cosmology but still maintaining the r,t dependence. Suppose indeed $\rho(r,t) = -p(r,t)$. From (15), (10) one has then

$$\rho(r,t) = -\lambda\dot{y}(r,t)/y(r,t) + \rho_0 \tag{16}$$

ρ_0 an arbitrary numerical constant. From (13), (14) one has

$$m(r,t) = -\rho_0 y^3/3, \qquad 3\,\dot{y}^2 + k\rho_0\, y^2 = 6\,E, \tag{17}$$

The last equation can be integrated. This is the only case we are able to furnish non factorized solutions. By distinguishing according to the signs of $E,\ k\rho_0$, one obtains

$$y(r,t) = \sqrt{\frac{6E}{k\rho_0}} \sin\left(\sqrt{\frac{k\rho_0}{3}}\, t + \alpha(r)\right), \quad k\rho_0 > 0,\ E > 0 \tag{18}$$

$$y(r,t) = \sqrt{\frac{6|E|}{|k\rho_0|}} \cosh\left(\sqrt{\frac{|k\rho_0|}{3}}\, t + \beta(r)\right), \quad k\rho_0 < 0,\ E < 0 \tag{19}$$

$$y(r,t) = \sqrt{\frac{6E}{|k\rho_0|}} \sinh\left(\sqrt{\frac{|k\rho_0|}{3}}\, t + \gamma(r)\right), \quad k\rho_0 < 0,\ E > 0 \tag{20}$$

$\alpha(r)$, $\beta(r)$, $\gamma(r)$ arbitrary integration functions. A time oscillating solution is then possible that starts with a big-bang $y(r,0) = 0$, if $\alpha = 0$. The solutions (19), (20) represent an indefinitely expanding Universe with non factorized physical radius accelerated expansion that may start with a big-bang $y(r,0) = 0$ if $\gamma = 0$ but that does not have an initial inflationary phase as in the RW case [13]. Here $p = p(r,t)$ and follows from (16) and assumptions.

5 The special case $p = p(t)$: variable separation

Suppose now $p = p(t)$. From (11) there follows that the physical radius has a factorized variable dependence $y = T(t)F(r)$. Using this relation in (5), (8) there follows that $\partial_r \rho = 0$ that is $\rho = \rho(t)$. Then equation (13) separates and (15) does not in fact depend on t. One has

$$F^2(r) \;=\; E(r)/H \tag{21}$$

$$3\dot{T}^2 + k\rho T^2 + k\lambda T\dot{T} = 6H \tag{22}$$

$$\dot{\rho} + 3(p+\rho)(\dot{T}/T) = -\lambda\partial_t(\dot{T}/T) \tag{23}$$

H the separation constant of (13). The r-dependence of y is therefore quite arbitrary and, by (21) and (8), it plays the role of a generalized Robertson-Walker coordinate. Instead the time dependence follows by solving the coupled equations (22), (23), once a state equation between ρ and p has been given.

5.1 The subcase $p(t) = w\rho(t)$

Suppose now $p(t) = w\rho(t),\ w \in \mathbf{R}$. From (22), (23) one has

$$\rho = (6H - 3\dot{T}^2 - k\lambda\dot{T}T)/(kT^2) \tag{24}$$

$$2T\ddot{T} + aT\dot{T} + b\dot{T}^2 = c \tag{25}$$

$$a = k\lambda(1+w), \quad b = (1+3w), \quad c = 2(1+3w)H \tag{26}$$

Given $T(t)$ solution of (25), $\rho(t)$ follows from (24). It seems of interest to solve the time equation because it contains the effect of particle creation. By setting $\eta = \dot{T}$, equation (25) is converted into

$$\eta\eta'_x = f(x)\eta^2 + g(x)\eta + h(x) \quad (x \equiv T) \tag{27}$$

$$f(x) = -d/2x, \quad g(x) = -a/2, \quad h(x) = c/2x \tag{28}$$

that falls into the class of Abel's non linear equation of second kind. In turn it can be canonically reported to the normal form [7]. Indeed by defining $\eta = E(x)u$, $E(x) \equiv \int f(x)dx = Ax^{-b/2}$, $\phi(x) = F_0(x)/F_1(x)$, $F_0(x) = h(x)/E^2(x)$, $F_1(x) = g(x)/E(x)$ and considering the variable z

$$z(x) = \int F_1(x)dx = -\frac{a}{A(b+2)}x^{(b+2)/2}, \tag{29}$$

the equation (27) assumes the normal form

$$uu'_z - u = \phi(x(z)) \equiv z^{(b-2)/(b+2)} \tag{30}$$

where the arbitrary integration constant A has been chosen to make 1 the factor in front of the last right term in (30). Once the solution $u = u(z)$ is given one has then

$$\eta \equiv \dot{T} = E(x)u(z(x)) = AT^{-b/2}\, u\Big(-\frac{aT^{(b+2)/2}}{A(b+2)}\Big) \tag{31}$$

Integrating by separation one finally obtains

$$-\frac{2}{a}\int \frac{dz}{u(z)} = t + B \tag{32}$$

B an integration constant. The solution of (25) can then be obtained in the form $F(z) \equiv F(z(T)) = t + B$ that in principle gives $T = T(t)$.

The result follows once two preliminary steps are possible: the solution of (30) is explicitly given and the calculation of the quadrature in (32) is performed. In the following it is shown that it is possible to perform at least the first step.

Recently the general solution of the Normal Abel's type non linear ODE of second kind has been given [4]. The solution depends on the value D of the discriminant of a suitable cubic equation. Accordingly, when applied to the present scheme, there results that if $D \neq 0$, the solutions take complex values and therefore are not acceptable. Instead if $D = 0$ there are real solutions u of (30) that result as follows:

$$\Big[u(z) - u_1(z)\Big]^{m_1(z)} \Big[u(z) - u_2(z)\Big]^{m_2(z)} = C \tag{33}$$

$$u_i(z) = \frac{z}{2}\Big[(2-3\delta_{i2})\Big(-\frac{q(z)}{2}\Big)^{\frac{1}{3}} + \frac{1}{3}\Big] \quad i = 1,2 \tag{34}$$

$$q(z) = -\frac{20}{27} + \frac{4}{3z}[\phi(z) - 2G(z)] \tag{35}$$

$$G(z) = -\phi(z) - \frac{324}{z^2}\phi^3(z) - \frac{54}{z}\phi^2(z) + \frac{z}{2} \quad (z \neq 0) \tag{36}$$

$$m_i(z) = M(z)\frac{z\delta_{i1} + 6(1+\delta_{i1})\phi(z)}{z + 18\phi(z)} \quad i = 1,2 \tag{37}$$

$$\frac{1}{M(z)} = \frac{4}{\log|C|}\int \frac{[z + 15\phi(z)]\log|4 + \frac{z}{3\phi(z)}|}{3[z + 12\phi(z)][z + 18\phi(z)]}dz \tag{38}$$

where C is a constant of integration, $u_i(z)$, $i = 1,2$ particular solutions of (7), δ_{ik} the Kronecker delta and $\phi(z)$ the expression given in (30). Therefore, even if quite cumbersome, the solution of (7) is mathematically completely defined.

Unfortunately, for what concerns the physical interpretation, the solution is far from being clear and explicit. Indeed one should first establish the z-region in order to obtain from (33) real $u(z)$, to perform the quadrature (32) and then to possibly obtain $T = T(t)$. Such steps seem however very involved. (The same holds, as far as the author is concerned, if you try to perform the quadrature (32) with the particular solutions $u_i(z)$ given in (34)). These considerations prevent to clearly establish whether the Universe has e. g., an inflationary phase and/or a late accelerated expansion. Therefore it is not yet possible to decide whether to accept or reject the cosmological model proposed in [14].

6 Comments and open problems

In the previous Sections the LTB cosmological model with particle creation previously studied [14] has been reconsidered with non trivial pressure. The solution of the scheme is not easy if one let ρ and p to depend on both r and t. Indeed for $p(r,t) = -\rho(r,t)$ (the Vacuum Energy case of the Standard Cosmology [3]) the general non factorized solution is determined but it remains open the problem of solving in the scheme for other general state equations.

The other situations discussed are all under the general assumption that the pressure depends only on the time t. This implies that also $\rho = \rho(t)$, that the physical radius has the form $y(r,t) = T(t)F(r)$. The scheme results completely separable. $F(r)$ can be formally viewed as a generalized Robertson-Walker coordinate while $T(t)$ satisfies a suitable non linear equation. Since the time equation contains the effect of the particle creation term it has been explicitly solved in case $p(t) = w\rho(t)$. The solution has been completely determined on mathematical ground but, as remarked, it seem difficult to put into evidence time aspects of physical interest.

The considerations of the present paper have been done at a qualitative level without deriving numerical results to be compared with cosmological data. The inclusion of particle creation effect of particle creation could be however of interest in an LTB cosmological model where inhomogeneity of the Universe is suggested as an alternative to the introduction of the notion of dark energy and dark matter as it is assumed in the standard cosmological model (e.g., [2, 1, 6]).

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