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Families of Sheffer Sequences Satisfying Generalizations of Power and Alternating Power Sum Identities

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Abstract

In this paper, we will consider one family of Sheffer sequences satisfying a generalization of the classical power sum identity. Also, we will study another family of Sheffer sequences satisfying a generalization of the classical alternating power sum identity.

1. Introduction

Let

$$\mathcal{F}=\left\{ f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1.1}$$

For $\mathbb{P}=\mathbb{C}[x]$, let us assume that $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L | p(x) \rangle$ denotes the action of the linear funcdtional L on $p(x)$ which satisfies $\langle L+M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$, and $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$, where c is a complex constant. The linear funcdtional $\langle f(t) | \cdot \rangle$ on $\mathbb{P}$ is defined by $\langle f(t) | x^n \rangle = a_n$, $(n \geq 0)$, for $f(t) \in \mathcal{F}$.

Thus, we have

$$\left\langle t^k \middle| x^n \right\rangle = n! \delta_{n,k}, \quad (n,k \geq 0), \quad (\text{see [14, 17]}), \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

The order $o(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t))=0$, then $f(t)$ is called an invertible series; if $o(f(t))=1$, then $f(t)$ is called a delta series (see [10, 17]).

Let us assume that $f_L(t)=\sum_{k=0}^{\infty} \left\langle L \middle| x^k \right\rangle \frac{t^k}{k!}$. From (1.2), we note that $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. Let $f(t)$, $g(t) \in \mathcal{F}$, with $o(f(t))=1$ and $o(g(t))=0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x)=n$) such that $\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n! \delta_{n,k}$, $(n,k \geq 0)$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (x \in \mathbb{C}), \quad (\text{see [15, 17]}), \tag{1.3}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f\left(\overline{f}(t)\right) = t$. Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then, by (1.2), we get

$$f(t)=\sum_{k=0}^{\infty} \left\langle f(t) \middle| x^k \right\rangle \frac{t^k}{k!}, \quad p(x)=\sum_{k=0}^{\infty} \left\langle t^k \middle| p(x) \right\rangle \frac{x^k}{k!}. \tag{1.4}$$

From (1.4), we can derive the following equations:

$$t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x), \quad e^{yt} p(x) = p(x+y), \quad \left\langle e^{yt} \middle| p(x) \right\rangle = p(y). \tag{1.5}$$

In this paper, we will consider one family of Sheffer sequences satisfying a generalization of the classical power sum identity. Also, we will study another family of Sheffer sequences satisfying a generalization of the classical alternating power sum identity. One family consists of those Sheffer sequences $s_n(x)$ for the pair $\left(g(t) = \frac{\left(e^{a_1 t}-1\right)\cdots\left(e^{a_r t}-1\right)}{f(t)^r}, f(t)\right)$, where $f(t)$ is any delta series, $r \in \mathbb{Z}_{>0}$, and $a_1, a_2, \ldots, a_r \neq 0$. Note that $g(t)$ is an invertible series. That is,

$$s_n(x) \sim \left(\frac{\left(e^{a_1 t}-1\right)\cdots\left(e^{a_r t}-1\right)}{f(t)^r}, f(t)\right). \tag{1.6}$$

We will show later that this family contains many interesting Sheffer sequences. Another family is composed of those Sheffer sequences $s_n(x)$ for the pair

$$\left(g(t) = \prod_{i=1}^{r}\left(\frac{e^{a_i t}+1}{2}\right), f(t)\right),$$

where $a_1, a_2, \ldots, a_r \neq 0$, and $f(t)$ is any delta series. Again, we will see that this family also has many interesting members.

In the previous paper ([10]) "A generalization of power and alternating power sums to any Appell polynomials", we introduced Barnes' multiple Bernoulli and Appell mixed-type polynomials and Barnes' multiple Euler and Appell mixed-type polynomials. Then we established one main identity for each of them connecting a sum for the Appell polynomial and that for the mixed-type polynomial. We note that the present result has overlaps with those ones only when $f(t) = t$.

2. Families of Sheffer sequences satisfying generalizations of power and alternating power sum identities

Let $a \neq 0$. From (1.5), we note that

$$\frac{e^{(m+1)at}-1}{e^{at}-1} p(x) = \sum_{i=0}^{m} p(x+ai), \quad \left\langle \frac{e^{(m+1)at}-1}{e^{at}-1} \middle| p(x) \right\rangle = \sum_{i=0}^{m} p(ai), \tag{2.1}$$

and

$$\begin{aligned} \frac{(-1)^m e^{(m+1)at}+1}{e^{at}+1} p(x) &= \sum_{i=0}^{m} (-1)^i p(x+ai), \\ \left\langle \frac{(-1)^m e^{(m+1)at}+1}{e^{at}+1} \middle| p(x) \right\rangle &= \sum_{i=0}^{m} (-1)^i p(ai), \end{aligned} \tag{2.2}$$

where $p(x)$ is any polynomial.

Lemma 1. *Let $m_1, m_2, \ldots, m_r \in \mathbb{Z}$ with $m_i \geq 0$ $(i = 1, 2, \ldots, r)$, $a_1, \ldots, a_r \in \mathbb{C} \setminus \{0\}$. Then, for any polynomial $p(x)$, we have*

$$\left(e^{(m_r+1)a_r t}-1\right)\cdots\left(e^{(m_1+1)a_1 t}-1\right) p(x) \tag{2.3}$$

$$=\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right).$$

Proof. We prove Lemma 1 by induction on r. It is easy to check that it holds for $r=1$.

Assume that, for $r>1$, the following holds:

$$\begin{aligned}&\left(e^{(m_{r-1}+1)a_{r-1}t}-1\right)\cdots\left(e^{(m_1+1)a_1t}-1\right)p(x)\\=&\sum_{i=0}^{r-1}(-1)^{r+i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right).\end{aligned}\tag{2.4}$$

Thus, by (2.4), we see that the LHS of (2.3) is

$$\begin{aligned}&\sum_{i=0}^{r-1}(-1)^{r-1-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}\left(e^{(m_r+1)a_rt}-1\right)p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right)\\
=&\sum_{i=0}^{r-1}(-1)^{r-1-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}\left(p\left(x+\sum_{j\in J}(m_j+1)\,a_j+(m_r+1)\,a_r\right)\right.\\
&\left.-p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right)\right)\\
=&\sum_{i=0}^{r-1}(-1)^{r-1-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j+(m_r+1)\,a_r\right)\\
&+\sum_{i=0}^{r-1}(-1)^{r-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right)\\
=&p\left(x+\sum_{j\in[1,r]}(m_j+1)\,a_j\right)\\
&+\sum_{i=1}^{r-1}(-1)^{r-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i-1}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j+(m_r+1)\,a_r\right)\\
&+\sum_{i=1}^{r-1}(-1)^{r-i}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)\,a_j\right)+(-1)^r p(x)\end{aligned}$$

$$
\begin{aligned}
&= p\left(x+\sum_{j\in[1,r]}(m_j+1)a_j\right)\\
&\quad+\sum_{i=1}^{r-1}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)a_j\right)+(-1)^r p(x)\\
&=\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}p\left(x+\sum_{j\in J}(m_j+1)a_j\right).
\end{aligned}
$$

□

Theorem 1. *Let* $s_n(x)\sim\left(\frac{(e^{a_1t}-1)\cdots(e^{a_rt}-1)}{f(t)^r},f(t)\right)$, $w_n(x)\sim(1,f(t))$, *where* $a_1,a_2,\ldots,a_r\in\mathbb{C}\setminus\{0\}$, $r\in\mathbb{Z}$ *with* $r>0$ *and* $o(f(t))=1$. *Then, we have*

$$
\begin{aligned}
&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}w_n(x+a_1i_1+\cdots+a_ri_r) &(2.5)\\
&=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}s_{n+r}\left(x+\sum_{j\in J}(m_j+1)a_j\right),
\end{aligned}
$$

where $(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l$ *and* $S_1(n,l)$ *is the Stirling number of the first kind.*

Proof. The result is obtained by computing the following in two different ways:

$$
\left(\frac{e^{(m_r+1)a_rt}-1}{e^{a_rt}-1}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1t}-1}{e^{a_1t}-1}\right)w_n(x). \tag{2.6}
$$

On one hand, it is

$$
\begin{aligned}
&\left(\frac{e^{(m_r+1)a_rt}-1}{e^{a_rt}-1}\right)\times\cdots\times\left(\frac{e^{(m_1+1)a_1t}-1}{e^{a_1t}-1}\right)w_n(x) &(2.7)\\
&=\frac{e^{(m_r+1)a_rt}-1}{e^{a_rt}-1}\cdots\frac{e^{(m_2+1)a_2t}-1}{e^{a_2t}-1}\left(\sum_{i_1=0}^{m_1}w_n(x+a_1i_1)\right)\\
&=\sum_{i_1=0}^{m_1}\frac{e^{(m_r+1)a_rt}-1}{e^{a_rt}-1}\cdots\frac{e^{(m_3+1)a_3t}-1}{e^{a_3t}-1}\left(\sum_{i_2=0}^{m_2}w_n(x+a_1i_1+a_2i_2)\right)\\
&=\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\frac{e^{(m_r+1)a_rt}-1}{e^{a_rt}-1}\cdots\frac{e^{(m_3+1)a_3t}-1}{e^{a_3t}-1}w_n(x+a_1i_1+a_2i_2).
\end{aligned}
$$

Continuing in this fashion, we obtain the expression on the LHS of (2.5).

On the other hand, we first observe that

$$\frac{(e^{a_1 t}-1)\cdots(e^{a_r t}-1)}{f(t)^r}s_n=w_n(x), f(t)w_n(x)=nw_{n-1}(x). \tag{2.8}$$

Thus, by (2.8), we get

$$\left(e^{a_1 t}-1\right)\cdots\left(e^{a_r t}-1\right)s_n(x)=f(t)^r w_n(x)=(n)_r w_{n-r}(x). \tag{2.9}$$

Replacing n by $n+r$, we have

$$\left(e^{a_1 t}-1\right)\cdots\left(e^{a_r t}-1\right)s_{n+r}(x)=(n+r)_r w_n(x). \tag{2.10}$$

From (2.6) and (2.10), we can derive the following equation:

$$\begin{aligned}&\frac{e^{(m_r+1)a_r t}-1}{e^{a_r t}-1}\cdots\frac{e^{(m_1+1)a_1 t}-1}{e^{a_1 t}-1}w_n(x) \qquad (2.11)\\ =&\frac{e^{(m_r+1)a_r t}-1}{e^{a_r t}-1}\cdots\frac{e^{(m_1+1)a_1 t}-1}{e^{a_1 t}-1}\left(\frac{1}{(n+r)_r}\prod_{l=1}^{r}\left(e^{a_l t}-1\right)s_{n+r}(x)\right)\\ =&\frac{1}{(n+r)_r}\left(e^{(m_r+1)a_r t}-1\right)\cdots\left(e^{(m_1+1)a_1 t}-1\right)s_{n+r}(x).\end{aligned}$$

Now, we get the expression on the RHS of (2.5) from Lemma 1. □

Corollary 1.

(a) Let $s_n(x)\sim\left(\frac{\left(e^{a_1 t}-1\right)\cdots\left(e^{a_r t}-1\right)}{f(t)^r}, f(t)\right)$, $w_n(x)\sim(1,f(t))$, *where* $a_1,a_2,\dots,a_r\in\mathbb{C}\setminus\{0\}$, $r\in\mathbb{Z}$ *with* $r>0$ *and* $o(f(t))=1$. *Then we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}w_n(a_1 i_1+\cdots+a_r i_r)=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}s_{n+r}\left(\sum_{j\in J}(m_j+1)a_j\right).$$

(b) Let $s_n(x)\sim\left(\left(\frac{e^t-1}{f(t)}\right)^r, f(t)\right)$, $w_n(x)\sim(1,f(t))$, *where* $o(f(t))=1$ *and* $r>0$. *Then*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}w_n(x+i_1+\cdots+i_r)=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}s_{n+r}\left(x+\sum_{j\in J}(m_j+1)\right).$$

(c) With $s_n(x), w_n(x)$ *as in (b), we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}w_n(i_1+\cdots+i_r)=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}s_{n+r}\left(\sum_{j\in J}(m_j+1)\right).$$

(d) With $s_n(x), w_n(x)$ *as in (b), we have*

$$\sum_{i_1,\cdots,i_r=0}^{m}w_n(x+i_1+\cdots+i_r)=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}s_{n+r}(x+(m+1)i).$$

(e) *With $s_n(x), w_n(x)$ as in (b), we have*

$$\sum_{i_1,\cdots,i_r=0}^{m} w_n(i_1+\cdots+i_r) = \frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}s_{n+r}((m+1)i).$$

Lemma 2. *Let $m_1, m_2, \ldots, m_r \in \mathbb{Z}$ with $m_i \geq 0$ $(i=1,2,\ldots,r)$, $a_1,\ldots,a_r \in \mathbb{C}\setminus\{0\}$. Then, for any polynomial $p(x)$, we have*

$$\begin{aligned}&\left((-1)^{m_r}e^{(m_r+1)a_rt}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1t}+1\right)p(x) \\ =&\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right),\end{aligned}\tag{2.12}$$

where $m_J=\sum_{j\in J}m_j$, $((m+1)a)_J=\sum_{j\in J}(m_j+1)a_j$.

Proof. We show this by induction on r. It is easy to check that it holds for $r=1$. Assume that, for $r>1$, the following holds:

$$\begin{aligned}&\left((-1)^{m_{r-1}}e^{(m_{r-1}+1)a_{r-1}t}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1t}+1\right)p(x) \\ =&\sum_{i=0}^{r-1}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right),\end{aligned}\tag{2.13}$$

From (2.12) and (2.13), we note that the LHS of (2.12) is

$$\begin{aligned}&\left((-1)^{m_r}e^{(m_r+1)a_rt}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1t}+1\right)p(x) \\ =&\sum_{i=0}^{r-1}\left((-1)^{m_r}e^{(m_r+1)a_rt}+1\right)\sum_{\substack{J\subset[1,r-1]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right) \\ =&\sum_{i=0}^{r-1}\left\{\sum_{\substack{J\subset[1,r-1]\\|J|=i}}(-1)^{m_J+m_r}p\left(x+((m+1)a)_J+(m_r+1)a_r\right)\right. \\ &\left.\sum_{\substack{J\subset[1,r-1]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right)\right\} \\ =&(-1)^{m_{[1,r]}}p\left(x+((m+1)a)_{[1,r]}\right) \\ &+\sum_{i=1}^{r-1}\sum_{\substack{J\subset[1,r-1]\\|J|=i-1}}(-1)^{m_J+m_r}p\left(x+((m+1)a)_J+(m_r+1)a_r\right)\end{aligned}\tag{2.14}$$

$$+\sum_{i=1}^{r-1}\sum_{\substack{J\subset[1,r-1]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right)+p(x)$$

$$=(-1)^{m_{[1,r]}}p\left(x+((m+1)a)_{[1,r]}\right)+\sum_{i=1}^{r-1}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right)+p(x)$$

$$=\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}p\left(x+((m+1)a)_J\right).$$

□

Theorem 2. *Let* $s_n(x)\sim\left(\prod_{i=1}^{r}\left(\frac{e^{a_it}+1}{2}\right),f(t)\right)$, $w_n(x)\sim(1,f(t))$, *where* $a_1,a_2,\dots,a_r\in\mathbb{C}\setminus\{0\}$, $r\in\mathbb{Z}$ *with* $r>0$ *and* $o(f(t))=1$. *Then we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}w_n(x+a_1i_1+\cdots+a_ri_r) \tag{2.15}$$
$$=\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}s_n\left(x+\sum_{j\in J}(m_j+1)a_j\right),$$

where $m_J=\sum_{j\in J}m_j$.

Proof. The result is obtained by computing the following in two different ways:

$$\frac{(-1)^{m_r}e^{(m_r+1)a_rt}+1}{e^{a_rt}+1}\cdots\frac{(-1)^{m_1}e^{(m_1+1)a_1t}+1}{e^{a_1t}+1}w_n(x). \tag{2.16}$$

On one hand, it is

$$\frac{(-1)^{m_r}e^{(m_r+1)a_rt}+1}{e^{a_rt}+1}\cdots\frac{(-1)^{m_2}e^{(m_2+1)a_2t}}{e^{a_2t}+1}\sum_{i_1=0}^{m_1}(-1)^{i_1}w_n(x+a_1i_1) \tag{2.17}$$
$$=\sum_{i_1=0}^{m_1}(-1)^{i_1}\frac{(-1)^{m_r}e^{(m_r+1)a_rt}+1}{e^{a_rt}+1}\cdots\frac{(-1)^{m_3}e^{(m_3+1)a_3t}}{e^{a_3t}+1}\frac{(-1)^{m_2}e^{(m_2+1)a_2t}}{e^{a_2t}+1}w_n(x+a_1i_1)$$
$$=\sum_{i_1=0}^{m_1}(-1)^{i_1}\frac{(-1)^{m_r}e^{(m_r+1)a_rt}+1}{e^{a_rt}+1}\cdots\frac{(-1)^{m_3}e^{(m_3+1)a_3t}}{e^{a_3t}+1}\left(\sum_{i_2=0}^{m_2}(-1)^{i_2}w_n(x+a_1i_1+a_2i_2)\right).$$

Continuing in this fashion, we get the expression on the LHS of (2.15).

On the other hand, (2.16) is

$$\frac{1}{2^r}\left((-1)^{m_r}e^{(m_r+1)a_rt}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1t}+1\right)\prod_{i=1}^{r}\left(\frac{2}{e^{a_it}+1}\right)w_n(x) \tag{2.18}$$

$$=\frac{1}{2^r}\left((-1)^{m_r}e^{(m_r+1)a_rt}+1\right)\cdots\left((-1)^{m_1}e^{(m_1+1)a_1t}+1\right)s_n(x).$$

Here we observe that

$$s_n(x)=\prod_{i=1}^{r}\left(\frac{2}{e^{a_it}+1}\right)w_n(x),$$

which follows from

$$\prod_{i=1}^{r}\left(\frac{e^{a_it}+1}{2}\right)s_n(x)=w_n(x)\sim(1,f(t)).$$

Now, we obtain the expresssion on the RHS of (2.15) from Lemma 2. □

Corollary 2.

(a) Let $s_n(x)\sim\left(\left(\frac{e^t+1}{2}\right)^r,f(t)\right)$, $w_n(x)\sim(1,f(t))$, *where* $f(t)$ *is a delta series. Then we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}w_n(x+i_1+\cdots+i_r)$$
$$=\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}s_n\left(x+\sum_{j\in J}(m_j+1)\right).$$

(b) With $s_n(x),w_n(x)$ *as in (a), we have*

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}w_n(i_1+\cdots+i_r)=\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}s_n\left(\sum_{j\in J}(m_j+1)\right).$$

(c) With $s_n(x),w_n(x)$ *as in (a), we have*

$$\sum_{i_1,\ldots,i_r=0}^{m}(-1)^{i_1+\cdots+i_r}w_n(x+i_1+\cdots+i_r)=\frac{1}{2^r}\sum_{i=0}^{r}(-1)^{mi}\binom{r}{i}s_n(x+(m+1)i).$$

(d) With $s_n(x),w_n(x)$ *as in (a), we have*

$$\sum_{i_1,\cdots,i_r=0}^{m}(-1)^{i_1+\cdots+i_r}w_n(i_1+\cdots+i_r)=\frac{1}{2^r}\sum_{i=0}^{r}(-1)^{mi}\binom{r}{i}s_n((m+1)i).$$

3. Examples on Theorem 1

(A) Let $s_n(x)=B_n(x\mid a_1,\ldots,a_r)\sim\left(\frac{(e^{a_1t}-1)\cdots(e^{a_rt}-1)}{t^r},t\right)$, $w_n(x)\sim(1,t)$.

Here $B_n(x|a_1,\ldots,a_r)$ are the Barnes' multiple Bernoulli polynomials whose generating function is given by

$$\frac{t^r}{(e^{a_1t}-1)\cdots(e^{a_rt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x\mid a_1,\ldots,a_r)\frac{t^n}{n!}.$$

From Theorem 1, we have

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}\left(x+a_1i_1+\cdots+a_ri_r\right)^n$$
$$=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}B_{n+r}\left(x+\sum_{j\in J}(m_j+1)\,a_j\,\middle|\,a_1,\ldots,a_r\right)..$$

Letting $x=0$, we also get

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}\left(a_1i_1+\cdots+a_ri_r\right)^n$$
$$=\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}B_{n+r}\left(\sum_{j\in J}(m_j+1)\,a_j\,\middle|\,a_1,\ldots,a_r\right).$$

For $r=1$, $a_1=1$, $m_1=m$, we have

$$\sum_{i=0}^{m}(x+i)^n=\frac{1}{n+1}\left(B_{n+1}(x+m)-B_{n+1}(x)\right).$$

Thus, for $x=0$, we get the classical power sum identity:

$$\sum_{i=0}^{m}i^n=\frac{1}{n+1}\left(B_{n+1}(m)-B_{n+1}\right).$$

(B) Let $s_n(x)=\beta_n^{(r)}(\lambda,x)\sim\left(\left(\frac{e^t-1}{\frac{1}{\lambda}\left(e^{\lambda t}-1\right)}\right)^r,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, $w_n(x)=(x\mid\lambda)_n=x(x-\lambda)\cdots(x-\lambda(n-1))\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$. Here, $\beta_n^{(r)}(\lambda,x)$ are the degenerate Bernoulli polynomials of order r whose generating function is given by

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n^{(r)}(\lambda,x)\frac{t^n}{n!}.$$

They are called the degenerate Bernoulli polynomials of order r, since

$$\lim_{\lambda\to0}\beta_n^{(r)}(\lambda,x)=B_n^{(r)}(x),\quad \lim_{\lambda\to\infty}\lambda^{-n}\beta_n^{(\lambda)}(\lambda,\lambda x)=b_n^{(r)}(x).$$

Here $B_n^{(r)}(x)$ are the Bernoulli polynomials of order r, with

$$\left(\frac{t}{e^t-1}\right)^r e^{tx}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [1–20])},$$

and $b_n^{(r)}(x)$ are the Bernoulli polynomials of the second kind of order r, with

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}b_n^{(r)}(x)\frac{t^n}{n!}.$$

From Corollary 1 (b), we have

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}\left(x+i_1+\cdots+i_r|\,\lambda\right)_n\\=&\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}\beta_{n+r}^{(r)}\left(\lambda,x+\sum_{j\in J}(m_j+1)\right).\end{aligned}$$

For $x=0$, we obtain

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}\left(i_1+\cdots+i_r|\,\lambda\right)_n\\=&m\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}\beta_{n+r}^{(r)}\left(\lambda,\sum_{j\in J}(m_j+1)\right),\end{aligned}$$

which reduces, in the simplest possible case, to

$$\sum_{i=0}^{m}(i\mid\lambda)_n=\frac{1}{n+1}\left(\beta_{n+1}(\lambda,m+1)-\beta_{n+1}(\lambda)\right). \qquad (*)$$

Here, $\beta_n(\lambda,x)=\beta_n^{(1)}(\lambda,x)$ were introduced by Carlitz in [3, 4] and the identity (*) was found also by Carlitz in [4]. Also, the higher-order degenerate Bernoulli polynomials $\beta_n^{(r)}(\lambda,x)$ were studied in [11] by using umbral calculus and in [13] by exploiting p-adic integrals.

(C) Let $s_n(x)=\beta_n(\lambda,x\mid a_1,\ldots,a_r)\sim\left(\frac{\left(e^{a_1t}-1\right)\cdots\left(e^{a_rt}-1\right)}{\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^r},\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, and $w_n(x)=(x\mid\lambda)_n\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$. We recall that $\beta_n(\lambda,x\mid a_1,\ldots,a_r)$ are called Barnes-type degenerate Bernoulli polynomials and studied with umbral calculus viewpoint in [14]. Here one shows easily that

$$\begin{aligned}\lim_{\lambda\to0}\beta_n(\lambda,x\mid a_1,\ldots,a_r)=&B_n(x\mid a_1,\ldots,a_r),\\ \lim_{\lambda\to\infty}\lambda^{-n}\beta_n(\lambda,\lambda x\mid a_1,\ldots,a_r)=&\left(\prod_{i=1}^{r}a_i\right)^{-1}b_n^{(r)}(x).\end{aligned}$$

From Theorem 1, we note that

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}\left(x+a_1i_1+\cdots+a_ri_r\mid\lambda\right)_n\\=&\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}\beta_{n+r}\left(\lambda,x+\sum_{j\in J}(m_j+1)a_j\,\middle|\,a_1,\ldots,a_r\right).\end{aligned}$$

(D) Let $s_n(x) \sim \left(\left(\frac{e^{2t}-1}{e^t-1}\right)^r = (e^t+1)^r, e^t-1\right)$, $w_n(x) = (x)_n \sim (1, e^t-1)$. Note that the generating function for $s_n(x)$ is

$$\left(\frac{1}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}.$$

Thus, we have $s_n(x) = 2^{-r}\mathrm{Ch}_n^{(r)}(x)$ where $\mathrm{Ch}_n^{(r)}(x)$ are the Changhee polynomials of the first kind of order r given by the generating function

$$\left(\frac{2}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(cf. [15])}.$$

Now, from Theorem 1, we obtain

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(x+2(i_1+\cdots+i_r))_n\\ =&\frac{1}{2^r(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}\mathrm{Ch}_{n+r}^{(r)}\left(x+2\sum_{j\in J}(m_j+1)\right).\end{aligned}$$

(E) Let $s_n(x) = (x)_n \sim \left(\left(\frac{e^t-1}{e^t-1}\right)^r = 1, e^t-1\right)$, $w_n(x) = (x)_n \sim (1, e^t-1)$. In this simple case, from Theorem 1, we have

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(x+i_1+\cdots+i_r)_n\\ =&\frac{1}{(n+r)_r}\sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{J\subset[1,r]\\|J|=i}}\left(x+\sum_{j\in J}(m_j+1)\right)_{n+r}.\end{aligned}$$

4. Examples on Theorem 2

(A) Let $s_n(x) = E_n(x \mid a_1,\ldots,a_r) \sim \left(\prod_{i=1}^r\left(\frac{e^{a_it}+1}{2}\right), t\right)$, $w_n(x) = x^n \sim (1,t)$. Here, $E_n(x \mid a_1,\ldots,a_r)$ are the Barnes-type Euler polynomials whose generating function is given by

$$\prod_{i=1}^{r}\left(\frac{2}{e^{a_it}+1}\right)e^{tx} = \sum_{n=0}^{\infty} E_n(x \mid a_1,\ldots,a_r)\frac{t^n}{n!}.$$

From Theorem 2, we obtain

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}(x+a_1i_1+\cdots+a_ri_r)^n$$

$$=\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}E_n\left(x+\sum_{j\in J}(m_j+1)\,a_j\mid a_1,\ldots,a_r\right).$$

For $r=1$, $a_1=1$, $m_1=m$, we have

$$\sum_{i=0}^{m}(-1)^i(x+i)^n=\frac{1}{2}\left((-1)^m E_n(x+m+1)+E_n(x)\right).$$

When $x=0$, we obtain the classical alternating power sum identity

$$\sum_{i=0}^{m}(-1)^i i^n=\frac{1}{2}\left((-1)^m E_n(m+1)+E_n\right). \tag{**}$$

(B) Let $s_n(x)=\mathcal{E}_n^{(r)}(\lambda,x)\sim\left(\left(\frac{e^t+1}{2}\right)^r,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, $w_n(x)=(x\mid\lambda)_n\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$. Here $\mathcal{E}_n^{(r)}(\lambda,x)$ are the degenerate Euler polynomials of order r whose generating function is given by

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(r)}(\lambda,x)\frac{t^n}{n!}.$$

These polynomials were studied in [12]. We observe that

$$\lim_{\lambda\to 0}\mathcal{E}_n^{(r)}(\lambda,x)=E_n^{(r)}(x),\quad \lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n^{(r)}(\lambda,\lambda x)=(x)_n\quad\text{for any } r\in\mathbb{Z}_{>0}.$$

Here $E_n^{(r)}(x)$ are the Euler polynomials of order r given by

$$\left(\frac{2}{e^t+1}\right)^r e^{tx}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}.$$

From Corollary 2 (a), we have

$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}(x+i_1+\cdots+i_r\mid\lambda)_n$$
$$=\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}\mathcal{E}_n^{(r)}\left(\lambda,x+\sum_{j\in J}(m_j+1)\right).$$

In the simplest possible case, this reduces to

$$\sum_{i=0}^{m}(-1)^i(i\mid\lambda)_n=\frac{1}{2}\left((-1)^m\mathcal{E}_n(\lambda,m+1)+\mathcal{E}_n(\lambda)\right),$$

which becomes, by letting $\lambda\to 0$, the classical alternating power sum identity (**). Higher order degenerate Euler polynomials $\mathcal{E}_n^{(r)}(\lambda,x)$ were introduced in [19] and studied in [12] by using umbral calculus.

(C) Let $s_n(x)=\mathcal{E}_n(\lambda,x\mid a_1,\ldots,a_r)\sim\left(\prod_{i=1}^{r}\left(\frac{e^{a_i t}+1}{2}\right),\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, $w_n(x)=$

$(x \mid \lambda)_n \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$. Here $\mathcal{E}_n(\lambda, x \mid a_1, \ldots, a_r)$ are the Barnes-type degenerate Euler polynomials given by

$$\prod_{i=1}^{r}\left(\frac{2}{(1+\lambda t)^{\frac{a_i}{\lambda}}+1}\right)(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda, x \mid a_1, \ldots, a_r)\frac{t^n}{n!},$$

which were studied in [9]. Here we note that

$$\lim_{\lambda\to 0}\mathcal{E}_n(\lambda, x \mid a_1, \ldots, a_r)=E_n(x \mid a_1, \ldots, a_r),$$
$$\lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n(\lambda, \lambda x \mid a_1, \ldots, a_r)=(x)_n.$$

From Theorem 2, we can derive the following equation:

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}(x+a_1i_1+\cdots+a_ri_r \mid \lambda)_n\\ =&\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{|J|\subset[1,r]\\|J|=i}}(-1)^{m_J}\mathcal{E}_n\left(\lambda, x+\sum_{j\in J}(m_j+1)a_j\,\middle|\, a_1, \ldots, a_r\right).\end{aligned}$$

(D) Let $s_n(x)=\mathrm{Ch}_n^{(r)}(x)\sim\left(\left(\frac{e^t+1}{2}\right)^r, e^t-1\right)$, $w_n(x)=(x)_n\sim(1, e^t-1)$. From Theorem 2, we note that

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}(x+a_1i_1+\cdots+a_ri_r)_n\\ =&\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}\mathrm{Ch}_n^{(r)}\left(x+\sum_{j\in J}(m_j+1)a_j\right).\end{aligned}$$

(E) Let $s_n(x)\sim\left(\left(\frac{e^t+1}{2}\right)^r, \log(1+t)\right)$, $w_n(x)\sim(1, \log(1+t))$. Here $s_n(x)$ are the polynomials whose generating function is

$$\begin{aligned}&\left(\frac{2}{e^{e^t-1}+1}\right)^r e^{x\left(e^t-1\right)}\\ =&\sum_{n=0}^{\infty}E_l^{(r)}(x)\frac{1}{l!}\left(e^t-1\right)^l\\ =&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_2(n,l)E_l^{(r)}(x)\right)\frac{t^n}{n!},\end{aligned}$$

where $S_2(n,l)$ is the Stirling number of the second kind.

Thus, $s_n(x)=\sum_{l=0}^{n}S_2(n,l)E_l^{(r)}(x)$.

Also, $w_n(x) = \phi_n(x) = \sum_{l=0}^{n} S_2(n,l)\, x^l$ are the exponential polynomials given by

$$e^{x\left(e^t-1\right)} = \sum_{n=0}^{\infty} \phi_n(x)\, \frac{t^n}{n!}.$$

From Corollary 2 (c), we have

$$\begin{aligned}&\sum_{i_1=0}^{m_1}\cdots\sum_{i_r=0}^{m_r}(-1)^{i_1+\cdots+i_r}\,\phi_n(x+i_1+\cdots+i_r)\\ =&\frac{1}{2^r}\sum_{i=0}^{r}\sum_{\substack{J\subset[1,r]\\|J|=i}}(-1)^{m_J}\sum_{l=0}^{n}S_2(n,l)\,E_l^{(r)}\left(x+\sum_{j\in J}(m_j+1)\right).\end{aligned}$$

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