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Solution of Neumann Boundary Value Problem for Electrical Field in Anisotropic Region at Hall Measurements

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Abstract

The paper suggests a method to solve Neumann boundary value problem with heterogeneous boundary conditions for electric field potential in relatively weak magnetic field in linear approximation. This occurs while measuring Hall effect with the help of probe methods. The expression for Hall field potential has been obtained, which is convenient for further practical use.

Keywords: potential, Neumann boundary value problem, Hall effect, Hall field, four-probe method

1. Introduction

In modern semiconductor electronics, materials that are characterized by anisotropic electrophysical properties are considered promising [1, 2]. Anisotropy of materials may be either natural - determined by crystal structure, or artificial - caused by the influence of external fields or deformations.

While theoretically grounding and developing probe methods of anisotropic material study one must solve electrodynamics boundary value problems (BVPs) [3, 4]. In particular, this necessity appears when one deals with probe measurements in magnetic fields while mathematically modeling electric fields. This task requires the solution of Neumann BVP [4].

The given work considers the method of boundary value electrodynamic problem applied to probe methods of measurement of anisotropic semiconductors parameters.

2. Boundary Value Problem Construction

Let us determine the potential distribution in galvanomagnetic phenomena in anisotropic rectangular semiconductors in case of probe measurement of the Hall coefficient (Fig.1). In the case considered, the rectangular sample is cut out so that its faces are parallel to crystallographic planes. The sample is placed in a transverse magnetic field **B,** a direct electric current $I_{(12)}$ (hereinafter, the subscripts in parentheses correspond to the contact numbers) passes through probes 1 and 2 (Fig.1). In this case, the electric-conductivity tensor is already nondiagonal [5]:

$$\hat{\sigma} = \begin{pmatrix} \sigma_x & \sigma_x \sigma_y R_z B & 0 \\ -\sigma_x \sigma_y R_z B & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \tag{1}$$

where σ_x, σ_y, σ_z are the components of the diagonal tensor of the electric conductivity with the magnetic field absent, R_z is a component of the coefficient Hall tensor.

Note that here the components of the electric-conductivity tensor depend linearly on the magnetic field induction **B**. This means that the Hall effect is considered in the region of relatively weak magnetic fields ($(\mu B)^2 << 1$), when the effect of the magnetic resistance that is determined by the B^2 – containing terms can be disregarded.

With no charge sources and drains we suppose [5]:

$$\operatorname{div} \mathbf{j} = 0, \tag{2}$$

where

$$\mathbf{j} = \hat{\sigma}\mathbf{E}, \ \mathbf{E} = -\mathrm{grad}\,\varphi . \tag{3}$$

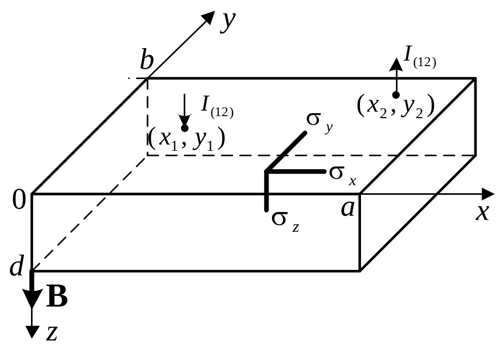


Fig.1. Schematic diagram of the arrangement of the current probes for anisotropic semiconductor placed in the transverse magnetic field. $I_{(12)}$ is probe current; (x_1, y_1), (x_2, y_2) are coordinates of probe centers*;* a , b , d – film size.

Consequently, according to formulae (1) – (3), electric field potential in the sample area satisfies the equation:

$$\sigma_x \frac{\partial^2 \varphi}{\partial x^2} + \sigma_y \frac{\partial^2 \varphi}{\partial y^2} + \sigma_z \frac{\partial^2 \varphi}{\partial z^2} = 0 . \tag{4}$$

Normal component of the current density vector on the surface of the sample under the study is different from zero only under current probes. According to Ohm's law (3) and the type of conductivity tensor (1) we obtain boundary conditions:

$$j_x\big|_{x=0,a} = -\left(\sigma_x \frac{\partial \varphi}{\partial x} + \sigma_x \sigma_y R_z B \frac{\partial \varphi}{\partial y} \right)\Bigg|_{x=0,a} = 0 , \tag{5}$$

$$j_y\big|_{y=0,b} = -\left(\sigma_y \frac{\partial \varphi}{\partial y} - \sigma_x \sigma_y R_z B \frac{\partial \varphi}{\partial x} \right)\Bigg|_{y=0,b} = 0 , \tag{6}$$

$$j_z\big|_{z=d} = -\sigma_z \frac{\partial \varphi}{\partial z}\bigg|_{z=d} = 0 , \tag{7}$$

$$j_z\big|_{z=0} = -\sigma_z \frac{\partial \varphi}{\partial z}\bigg|_{z=0} = -I_{(12)} \cdot \left[\delta\left(x - x_{(1)}\right) \cdot \delta\left(y - y_{(1)}\right) - \delta\left(x - x_{(2)}\right) \cdot \delta\left(y - y_{(2)}\right)\right], \quad (8)$$

where **j** is current density vector; *a, b, d* – length, width and thickness of the sample, $\delta(y)$, $\delta(x)$ — Dirac delta function, the usage of which is justified only for the current probes with small entry sectional area.

3. Method of Solution

It should be noted that the presented boundary value problem (4) – (8) doesn't belong to typical Dirichlet or Neumann BVPs. At present there's no exact mathematical solution to this problem. However, because the assumption of the linearity of the potential φ with respect to B was used, its solution can be represented in the form of a linear approximation with respect to the magnetic field. Thus, the desired potential can be represented as

$$\varphi = \varphi_0 + \varphi_H , \quad (9)$$

where φ_0 – is the potential of the electric field in the absence of an external magnetic field, φ_H – is the potential of the Hall field, which arises in the sample after switching the external magnetic field on. In this case, each of field components (9) must satisfy equation (4).

Substituting solution (9) into BVP (4) – (8), we obtain the corresponding BVP for the potential φ_0:

$$\sigma_x \frac{\partial^2 \varphi_0}{\partial x^2} + \sigma_y \frac{\partial^2 \varphi_0}{\partial y^2} + \sigma_z \frac{\partial^2 \varphi_0}{\partial z^2} = 0 , \quad (10)$$

$$\sigma_x \frac{\partial \varphi_0}{\partial x}\bigg|_{x=0,a} = 0 , \; \sigma_y \frac{\partial \varphi_0}{\partial y}\bigg|_{y=0,b} = 0 , \quad (11)$$

$$\sigma_z \frac{\partial \varphi_0}{\partial z}\bigg|_{z=d} = 0 , \; \sigma_z \frac{\partial \varphi_0}{\partial z}\bigg|_{z=0} = -I_{(12)} \cdot \left[\delta\left(x - x_{(1)}\right) \cdot \delta\left(y - y_{(1)}\right) - \delta\left(x - x_{(2)}\right) \cdot \delta\left(y - y_{(2)}\right)\right]; \quad (12)$$

and for the potential φ_H:

$$\sigma_x \frac{\partial^2 \varphi_H}{\partial x^2} + \sigma_y \frac{\partial^2 \varphi_H}{\partial y^2} + \sigma_z \frac{\partial^2 \varphi_H}{\partial z^2} = 0 , \quad (13)$$

$$\left(\sigma_x \frac{\partial \varphi_H}{\partial x} + \sigma_x \sigma_y R_z B \frac{\partial(\varphi_0 + \varphi_H)}{\partial y}\right)\Bigg|_{x=0,a} = 0, \quad \left(\sigma_y \frac{\partial \varphi_H}{\partial y} - \sigma_x \sigma_y R_z B \frac{\partial(\varphi_0 + \varphi_H)}{\partial x}\right)\Bigg|_{y=0,a} = 0, \quad (14)$$

$$\sigma_z \frac{\partial \varphi_H}{\partial z}\Bigg|_{z=0,d} = 0. \quad (15)$$

The solution of the BVP (10) – (12) has been described several times in literature [3, 6, 7]. That is why we present here only the main stages of its solution. Let us represent the general solution of equation (10) in the form of the trigonometric Fourier series

$$\varphi_0(x, y, z) = \sum_{n,k=0,1,\ldots} Z_{nk}(z)\cos(\alpha_n x)\cos(\beta_k y), \quad (16)$$

where

$$\alpha_k = \frac{\pi k}{a}, \qquad \beta_n = \frac{\pi n}{b}. \quad (17)$$

After substitution (16) into (10) we obtain the equation for the function $Z_{nk}(z)$::

$$\frac{\partial^2 Z_{nk}}{\partial z^2} - \eta_{nk}^2 Z_{nk} = 0; \qquad \eta_{nk}^2 = \frac{\sigma_x}{\sigma_z}\alpha_k^2 + \frac{\sigma_y}{\sigma_z}\beta_n^2. \quad (18)$$

Solution of the equation (18) may be presented as the sum of hyperbolic functions

$$Z_{nk}(z) = C_{nk} \cdot \sinh(\eta_{nk} z) + D_{nk} \cdot \cosh(\eta_{nk} z). \quad (19)$$

Coefficients C_{nk} and D_{nk} are obtained by mean of substituting boundary conditions (12) into function (16). Omitting the very cumbersome procedure of solving, let us just produce the final expression for φ_0

$$\varphi_0 = -\frac{4 I_{(12)}}{\sigma_z ab} \sum_{n,k=0}^{\infty} A_{nk} \cdot \frac{\cosh \eta_{kn}(d - z)}{\alpha_{kn} \sinh \eta_{kn} d} \cdot \cos \alpha_k x \cdot \cos \beta_n y, \quad (20)$$

where

$$A_{nk} = \begin{cases} \Theta_n \Theta_k \left(\cos(\alpha_k x_{(1)}) \cdot \cos(\beta_n y_{(1)}) - \cos(\alpha_k x_{(2)}) \cdot \cos(\beta_n y_{(2)})\right), & \Theta_{i\{i=k,n\}} = \begin{cases} 1, & i \neq 0; \\ 1/2, & i = 0; \end{cases} \\ 0, \text{ for } n = k = 0. \end{cases} \quad (21)$$

Solution of the boundary value problem (13) – (15) causes certain difficulties and isn't presented in the scientific literature. But, we have managed to work out the technique, which makes it possible to solve the given problem in the region of relatively weak magnetic fields. In boundary conditions (14) due to the condition used, we disregard summands $\sigma_x\sigma_y R_z B\frac{\partial\varphi_H}{\partial y}$ and $\sigma_x\sigma_y R_z B\frac{\partial\varphi_H}{\partial x}$, which are proportional to B^2 (as B^2 is dependant linearly on the magnetic strength B) [5].

Thus, boundary conditions (14) for potential of the Hall field acquire the form:

$$\left(\frac{\partial\varphi_H}{\partial x}+\sigma_y R_z B\frac{\partial\varphi_0}{\partial y}\right)\bigg|_{x=0,a}=0,\ \left(\frac{\partial\varphi_H}{\partial y}-\sigma_x R_z B\frac{\partial\varphi_0}{\partial x}\right)\bigg|_{y=0,a}=0,\ \frac{\partial\varphi_H}{\partial z}\bigg|_{z=0,d}=0. \qquad (22)$$

The solution of the equation (13) can be represented as

$$\varphi_H=U_H+V_H,$$

where boundary value problems for U_H and V_H correspondingly acquire the form:

$$\sigma_x\frac{\partial^2 U_H}{\partial x^2}+\sigma_y\frac{\partial^2 U_H}{\partial y^2}+\sigma_z\frac{\partial^2 U_H}{\partial z^2}=0, \qquad (23)$$

$$\sigma_x\frac{\partial U_H}{\partial x}\bigg|_{x=0,a}=0, \qquad \sigma_z\frac{\partial U_H}{\partial z}\bigg|_{z=0,d}=0, \qquad (24)$$

$$\sigma_y\frac{\partial U_H}{\partial y}\bigg|_{y=0,b}=\sigma_x\sigma_y R_z B\frac{\partial\varphi_0}{\partial x}\bigg|_{y=0,b}; \qquad (25)$$

$$\sigma_x\frac{\partial^2 V_H}{\partial x^2}+\sigma_y\frac{\partial^2 V_H}{\partial y^2}+\sigma_z\frac{\partial^2 V_H}{\partial z^2}=0, \qquad (26)$$

$$\sigma_y\frac{\partial V_H}{\partial y}\bigg|_{y=0,b}=0, \qquad \sigma_z\frac{\partial V_H}{\partial z}\bigg|_{z=0,d}=0, \qquad (27)$$

$$\sigma_x\frac{\partial V_H}{\partial x}\bigg|_{x=0,a}=-\sigma_x\sigma_y R_z B\frac{\partial\varphi_0}{\partial y}\bigg|_{x=0,a}. \qquad (28)$$

Let us solve boundary value problems (23) – (25) and (26) – (28) by the Fourier method, analogous to the solution of problem (10) – (12). After finding the expressions for functions U_H and V_H we obtain the expression for φ_H :

$$\varphi_H = \frac{16 I_{(12)}\sigma_x R_z B}{\sigma_z a^2 bd} \sum_{r,s,k,n=0}^{\infty} \Theta_r \Theta_s A_{nk} \frac{\left(1-(-1)^{n+r}\right)\beta_n^2}{\left(\beta_n^2-\beta_r^2\right)\xi_{rs}\left(\eta_{kn}^2+\delta_s^2\right)} \times$$
$$\times \frac{(-1)^k \cosh(\xi_{rs}x) - \cosh \xi_{rs}(x-a)}{\sinh(\xi_{rs}a)} \cos(\beta_r y)\cos(\delta_s z) -$$
$$-\frac{16 I_{(12)}\sigma_y R_z B}{\sigma_z ab^2 d} \sum_{p,q,k,n=0}^{\infty} \Theta_p \Theta_q A_{nk} \frac{\left(1-(-1)^{k+p}\right)\alpha_k^2}{\left(\alpha_k^2-\alpha_p^2\right)\omega_{pq}\left(\eta_{kn}^2+\delta_q^2\right)} \times$$
$$\frac{(-1)^n \cosh(\omega_{pq}y) - \cosh\left(\omega_{pq}(y-b)\right)}{\sinh(\omega_{pq}b)} \cos(\alpha_p x)\cos(\delta_q z), \tag{29}$$

where A_{nk} is determined by the expression (21) and

$$\alpha_p = \pi p/a,\ \beta_r = \pi r/b,\ \delta_q = \pi q/d,\ \delta_s = \pi s/d, \tag{30}$$

$$\omega_{pq}^2 = \left(\sigma_x\alpha_p^2 + \sigma_z\delta_q^2\right)/\sigma_y,\ \xi_{rs}^2 = \left(\sigma_y\beta_r^2 + \sigma_z\delta_s^2\right)/\sigma_x,\ \Theta_{i\{i=p,q,r,s\}} = \begin{cases}1, & i \neq 0;\\ 1/2, & i = 0.\end{cases} \tag{31}$$

Note that the index pairs (n, r) and (k, p) mustn't acquire simultaneously the same expression: $n \neq r$, $k \neq p$.

4. Analysis of the Obtained Solution and Practical Recommendations

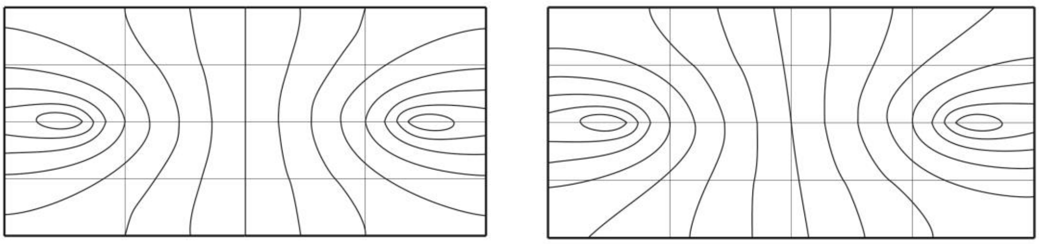
Obtained solutions (20) and (29) make it possible to analyze current potential distribution and current density in anisotropic samples placed in external magnetic field. These solutions also enable us to offer theoretically grounded techniques for measurement of the component of tensor electrical conductivity and Hall coefficient [3, 6, 7]. Obviously, these formulae are hard to put to practice in the form they are presented above, that is why, as a rule, in practice approximation of thin samples is used, when the distance between current contacts 1and 2 is much bigger than thickness *d*. It should be noted that in practice of probe methods measurements four-probe test device is used, which has a rigorously defined distance between current and measurement probes, plus, all measurements are carried out on the surface of the plate, that is at $z = 0$. Taking this into account, let's rigorously define the coordinates of the current probes: $x_{(1)} = a_0 - l_1, y_{(1)} = b_0$, $x_{(2)} = a_0 + l_1, y_{(2)} = b_0$, where (a_0, b_0) are coordinates of the test device, is the distance from the center of the test device to the current probe. Then the expression (29) for φ_H acquires a simpler form:

$$\varphi_H = \frac{16 I_{(12)} \sqrt{\sigma_x \sigma_y} R_z B}{\sigma_z a^2 b d} \sum_{\substack{n,\, p=1,2,\ldots \\ k=0,1,\ldots}} \left[\Theta_k \frac{\left[1-(-1)^{n+p}\right] \alpha_k^2 \sin(\alpha_k a_0) \cos(\beta_n b_0) \sin(\alpha_k l_1)}{\alpha_p \eta_{nk}^2 \left(\alpha_k^2 - \alpha_p^2\right)} \times \right.$$
$$\left. \times \frac{\cosh[\alpha_p \gamma (y-b)] - (-1)^k \cosh(\alpha_p \gamma y)}{\sinh(\alpha_p \gamma b)} \cdot \cos(\alpha_p x) \right] +$$
$$+ \frac{16 I_{(12)} \sqrt{\sigma_x \sigma_y} R_z B}{\sigma_z a b^2 d} \sum_{n,k,r=1,2,\ldots} \left[\frac{\left[1-(-1)^{k+r}\right] \beta_n^2 \sin(\alpha_k a_0) \cos(\beta_n b_0) \sin(\alpha_k l_1)}{\beta_r \eta_{nk}^2 \left(\beta_n^2 - \beta_r^2\right)} \times \right.$$
$$\left. \times \frac{(-1)^n \cosh(\beta_r x/\gamma) - \cosh[\beta_r (a-x)/\gamma]}{\sinh(\beta_r a/\gamma)} \cdot \cos(\beta_r y) \right], \qquad (32)$$

where $\gamma = \sqrt{\sigma_x / \sigma_y}$ is the parameter of conductivity anisotropy.

It should be noted that for practical calculation of the potential it's necessary to take not less than 100 terms of series for each of summable indices (*n*, *k*, *p*, *r*) with measurement error no more than 2-3%.

Fig.2 represents the equipotential lines built by expression (20) – without magnetic field (Fig. 2, a) and by expressions (9), (20), (32) with the magnetic field present (Fig. 2, b). While modeling the following parameters were used: $a/b = 2$, $d/a = 0.01$, $a_0 = a$, $b_0 = b$, $l_1 = 0.75a$, $\sigma_y = \sigma_z = 0.25\sigma_x$, $\sqrt{\sigma_x \sigma_y} R_z B = 0.2$. The given results demonstrate that the effect of magnetic field produces potential difference between symmetrical points on opposite sides of the sample (Hall effect voltage).



a) b)

Fig. 2. Models of the electric field in thin anisotropic samples (a – $B = 0$; b – $B \neq 0$).

The obtained expression (32) enable us to offer techniques for experimental assessment of Hall electromotive force at probe measurements, the practical realization of which doesn't require having solder contacts on the edge of the samples. It, in its turn, speeds up the process of semiconductor kinetic parameter measurements. It should also be mentioned that one needs modern computers to apply the obtained expression to practical usage; computers are necessary to calculate series in expressions (20) и (32). So, on the basis of the solution of boundary value electrodynamic problem we've obtained the expression for Hall field potential in linear approximation. The expression (32) for field potential is represented in the form convenient for further calculating Hall electromotive force.

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