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Symmetric Identities of the q-Euler Polynomials

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Abstract

In this paper, we study some symmetric identities of q-Euler numbers and polynomials. From these properties, we derive several identities of q-Euler numbers and polynomials.

1 Introduction

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},\ (see\ [2-6]). \tag{1.1}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$.

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. Let $q \in \mathbb{C}$ with $|q| < 1$. For any complex number x, the q-analogue of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to1}[x]_q = x$. Recently, T. Kim introduced a q-extension of Euler polynomials as follows:

$$F_q(t,x) = [2]_q \sum_{n=0}^{\infty}(-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!},\ (see[7,8]). \tag{1.2}$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the q-Euler numbers. From (1.2), we note that

$$\begin{aligned} E_{n,q}(x) &= (q^x E_q + [x]_q)^n \\ &= \sum_{l=0}^{n}\binom{n}{l} q^{xl} E_{l,q}[x]_q^{n-l},\ (see\ [7,8]), \end{aligned} \tag{1.3}$$

with the usual convention about replacing E_q^l by $E_{l,q}$.

In [8], Kim introduced q-Euler zeta function as follows:

$$\begin{aligned} \zeta_{E,q}(s,x) &= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_q(-t,x)dt \\ &= [2]_q\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{[n+x]_q^s}, \end{aligned} \tag{1.4}$$

where $x \neq 0, -1, -2, \ldots$, and $s \in \mathbb{C}$.

From (1.4), we have

$$\zeta_{E,q}(-m,x) = E_{m,q}(x), \tag{1.5}$$

where $m \in \mathbb{Z}_{\geq 0}$.

Recently, Y. Simsek gave recurrence symmetric identities for (h, q)-Euler polynomials and the alternating sums of powers of consecutive (h, q)-integers(see [9]) and Y. He gave some interesting symmetric identities of Carlitz's q-Bernoulli numbers and polynomials(see [1]). In this paper, we study some new symmetries of the q-Euler numbers and polynomials, which is the answer to an open question for the symmetric identities of Carlitz's type q-Euler numbers and polynomials in [5]. By using our symmetries for the q-Euler polynomials we can obtain some identities between q-Euler numbers and polynomials.

2 Symmetric identities of q-Euler polynomials

In this section, we assume that $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$. First, we observe that

$$\begin{aligned} \frac{1}{[2]_{q^a}}\zeta_{E,q^a}(s, bx + \frac{bj}{a}) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{[n + bx + \frac{bj}{a}]_{q^a}^s} \\ &= \sum_{n=0}^{\infty} \frac{q^{an}(-1)^n [a]_q^s}{[bj + abx + an]_q^s} = [a]_q^s \sum_{n=0}^{\infty}\sum_{i=0}^{b-1} \frac{(-1)^{i+bn} q^{a(i+bn)}}{[ab(x+n) + bj + ai]_q^s}. \end{aligned} \tag{2.1}$$

Thus, by (2.1), we get

$$\frac{[b]_q^s}{[2]_{q^a}} \sum_{j=0}^{a-1} (-1)^j q^{bj} \zeta_{E,q^a}(s, bx + \frac{bj}{a}) = [b]_q^s [a]_q^s \sum_{j=0}^{a-1}\sum_{i=0}^{b-1}\sum_{n=0}^{\infty} \frac{q^{ai+bj+abn}(-1)^{i+n+j}}{[ab(x+n) + bj + ai]_q^s}. \tag{2.2}$$

By the same method as (2.2), we get

$$\begin{aligned} &\frac{[a]_q^s}{[2]_{q^b}} \sum_{j=0}^{b-1} (-1)^j q^{aj} \zeta_{E,q^b}(s, ax + \frac{aj}{b}) \\ &= [a]_q^s [b]_q^s \sum_{j=0}^{b-1}\sum_{i=0}^{a-1}\sum_{n=0}^{\infty} \frac{q^{bi+aj+abn}(-1)^{i+n+j}}{[ab(x+n) + aj + bi]_q^s}. \end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $a, b \in \mathbb{N}$ with $a \equiv 1 \ (mod\ 2)$, $b \equiv 1 \ (mod\ 2)$,*

$$[2]_{q^b}[b]_q^s \sum_{j=0}^{a-1} (-1)^j q^{bj} \zeta_{E,q^a}(s, bx + \frac{bj}{a}) = [2]_{q^a}[a]_q^s \sum_{j=0}^{b-1} (-1)^j q^{aj} \zeta_{E,q^b}(s, ax + \frac{aj}{b}).$$

By (1.5) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{N}$ with $a \equiv 1$ (mod 2), $b \equiv 1$ (mod 2), we have*

$$[2]_{q^b}[a]_q^n \sum_{j=0}^{a-1} (-1)^j q^{bj} E_{n,q^a}(bx + \frac{bj}{a}) = [2]_{q^a}[b]_q^n \sum_{j=0}^{b-1} (-1)^j q^{aj} E_{n,q^b}(ax + \frac{aj}{b}).$$

From (1.3), we note that

$$\begin{aligned} E_{n,q}(x+y) &= (q^{x+y}E_q + [x+y]_q)^n \\ &= (q^{x+y}E_q + q^x[y]_q + [x]_q)^n \\ &= (q^x(q^yE_q + [y]_q) + [x]_q)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} q^{xi}(q^yE_q + [y]_q)^i [x]_q^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} q^{xi} E_{i,q}(y)[x]_q^{n-i}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following proposition.

Proposition 2.3. *For $n \geq 0$, we have*

$$\begin{aligned} E_{n,q}(x+y) &= \sum_{i=0}^{n} \binom{n}{i} q^{xi} E_{i,q}(y)[x]_q^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)x} E_{n-i,q}(y)[x]_q^{i}. \end{aligned}$$

Now, we observe that

$$
\begin{aligned}
&\sum_{j=0}^{a-1}(-1)^j q^{bj} E_{n,q^a}(bx+\frac{bj}{a}) \\
&=\sum_{j=0}^{a-1}(-1)^j q^{bj} \sum_{i=0}^{n} \binom{n}{i} q^{ia(\frac{bj}{a})} E_{i,q^a}(bx) \left[\frac{bj}{a}\right]_{q^a}^{n-i} \\
&=\sum_{j=0}^{a-1}(-1)^j q^{bj} \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)bj} E_{n-i,q^a}(bx) \left[\frac{bj}{a}\right]_{q^a}^{i} \qquad (2.5)\\
&=\sum_{i=0}^{n} \binom{n}{i} \left(\frac{[b]_q}{[a]_q}\right)^i E_{n-i,q^a}(bx) \sum_{j=0}^{a-1}(-1)^j q^{bj(n+1-i)} [j]_{q^b}^i \\
&=\sum_{i=0}^{n} \binom{n}{i} \left(\frac{[b]_q}{[a]_q}\right)^i E_{n-i,q^a}(bx) S^*_{n,i,q^b}(a),
\end{aligned}
$$

where $S^*_{n,i,q}(a)=\sum_{j=0}^{a-1}(-1)^j q^{(n+1-i)j}[j]_q^i$.

From (2.5), we can derive

$$
[2]_{q^b}[a]_q^n \sum_{j=0}^{a-1}(-1)^j q^{bj} E_{n,q^a}(bx+\frac{bj}{a}) = [2]_{q^b} \sum_{i=0}^{n} \binom{n}{i} [a]_q^{n-i}[b]_q^i E_{n-i,q^a}(bx) S^*_{n,i,q^b}(a). \qquad (2.6)
$$

By the same method as (2.6), we get

$$
[2]_{q^a}[b]_q^n \sum_{j=0}^{b-1}(-1)^j q^{aj} E_{n,q^b}(ax+\frac{aj}{b}) = [2]_{q^a} \sum_{i=0}^{n} \binom{n}{i} [b]_q^{n-i}[a]_q^i E_{n-i,q^b}(ax) S^*_{n,i,q^a}(b). \qquad (2.7)
$$

Therefore, by Theorem 2.2, (2.6) and (2.7), we obtain the following theorem.

Theorem 2.4. *For $n\in\mathbb{Z}_{\geq 0}$ and $a,b\in\mathbb{N}$ with $a\equiv 1\ (mod\, 2)$, $b\equiv 1\ (mod\, 2)$, we have*

$$
[2]_{q^b} \sum_{i=0}^{n} \binom{n}{i} [a]_q^{n-i}[b]_q^i E_{n-i,q^a}(bx) S^*_{n,i,q^b}(a) = [2]_{q^a} \sum_{i=0}^{n} \binom{n}{i} [b]_q^{n-i}[a]_q^i E_{n-i,q^b}(ax) S^*_{n,i,q^a}(b),
$$

where $S^*_{n,i,q}(a)=\sum_{j=0}^{a-1}(-1)^j q^{(n+1-i)j}[j]_q^i$.

It is easy to show that

$$[x]_q u + q^x [y+m]_q (u+v) = [x+y+m]_q (u+v) - [x]_q v. \tag{2.8}$$

Thus, by (2.8), we get

$$e^{[x]_q u} \sum_{m=0}^{\infty} q^m (-1)^m e^{[y+m]_q q^x (u+v)} = e^{-[x]_q v} \sum_{m=0}^{\infty} q^m (-1)^m q^{[x+y+m]_q (u+v)}. \tag{2.9}$$

The left hand side of (2.9) multiplied by $[2]_q$ is given by

$$\begin{aligned}
&[2]_q e^{[x]_q u} \sum_{m=0}^{\infty} q^m (-1)^m e^{[y+m]_q q^x (u+v)} \\
&= e^{[x]_q u} \sum_{n=0}^{\infty} q^{nx} E_{n,q}(y) \frac{(u+v)^n}{n!} \\
&= \left(\sum_{l=0}^{\infty} [x]_q^l \frac{u^l}{l!} \right) \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q^{(k+n)x} E_{k+n,q}(y) \frac{u^k}{k!} \frac{v^n}{n!} \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} q^{(k+n)x} E_{k+n,q}(y) [x]_q^{m-k} \right) \frac{u^m}{m!} \frac{v^n}{n!}.
\end{aligned} \tag{2.10}$$

The right hand side of (2.9) multiplied by $[2]_q$ is given by

$$\begin{aligned}
&[2]_q e^{-[x]_q v} \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+y+m]_q (u+v)} \\
&= e^{-[x]_q v} \sum_{n=0}^{\infty} E_{n,q}(x+y) \frac{(u+v)^n}{n!} \\
&= \left(\sum_{l=0}^{\infty} \frac{(-[x]_q)^l}{l!} v^l \right) \left(\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} E_{m+k,q}(x+y) \frac{u^m}{m!} \frac{v^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_{m+k,q}(x+y) (-[x]_q)^{n-k} \right) \frac{u^m}{m!} \frac{v^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_{m+k,q}(x+y) q^{(n-k)x} [-x]_q^{n-k} \right) \frac{u^m}{m!} \frac{v^n}{n!}.
\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we get

$$\sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} E_{n+k,q}(y) [x]_q^{m-k} = \sum_{k=0}^{n} \binom{n}{k} q^{(n-k)x} E_{m+k,q}(x+y) [-x]_q^{n-k} \tag{2.12}$$

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