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Generating Functions of the Generalized q-Genocchi Numbers and Polynomials with Weak Weight α

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Abstract

The purpose of this paper is to construct a new type of the generalized q-Genocchi numbers $\widetilde{G}_{n,\chi,q}^{(\alpha)}$ and polynomials $\widetilde{G}_{n,\chi,q}^{(\alpha)}(x)$. We give generating functions of the generalized q-Genocchi numbers and polynomials with weak weight α. Some interesting results and relationships are obtained.

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1 Introduction

p-adic analysis with q-analysis includes several domains in mathematics and physics, including the number theory, algebraic geometry, algebraic topology, mathematical analysis, mathematical physics, string theory, field theory, differential equations, partial differential equations, stochastic differential equations, quantum groups, dynamical systems, and other parts of the natural sciences. In particular, in recent years p-adic analysis and non-Archimedean functional analysis has been developed with its applications in mathematical physics. The Genocchi numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. The zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. As a consequence, many authors have studied the q-extension in various areas (see [1-9]). In this paper we construct the generalized q-Genocchi numbers $\widetilde{G}^{(\alpha)}_{n,\chi,q}$ and polynomials $\widetilde{G}^{(\alpha)}_{n,\chi,q}(x)$ with weak weight α. We also find generating functions of the generalized q-Genocchi numbers $\widetilde{G}^{(\alpha)}_{n,\chi,q}$ and polynomials $\widetilde{G}^{(\alpha)}_{n,\chi,q}(x)$ with weak weight α. By using the generalized q-Genocchi numbers $\widetilde{G}^{(\alpha)}_{n,\chi,q}$ and polynomials $\widetilde{G}^{(\alpha)}_{n,\chi,q}(x)$ with weak weight α, we construct the l-series and two variable l-series.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [4] .} \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{1.2}$$

Let a fixed positive integer d with $(p,d) = 1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p, , \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$I_{-q}(g) = \int_X g(x) d\mu_{-q}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x). \tag{1.3}$$

The Genocchi numbers G_n are defined by the generating function:

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}), \text{ cf. [5, 6, 7]} \tag{1.4}$$

where we use the technique method notation by replacing G^n by $G_n (n \geq 0)$ symbolically. We consider the Genocchi polynomials $G_n(x)$ as follows:

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{1.5}$$

2 Generalized q-Genocchi polynomais

In this section, our goal is to give generating functions of the generalized q-Genocchi numbers and polynomials with weak weight α. These numbers will be used to prove the analytic continuation of the l-series. Let q be a complex number with $|q| < 1$. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$. Then the generalized q-Genocchi numbers associated with associated with χ, $\widetilde{G}^{(\alpha)}_{n,\chi,q}$, are defined by the following generating function

$$\mathcal{G}^{(\alpha)}_{\chi,q}(t) = \frac{[2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d}e^{dt}+1} = \sum_{n=0}^{\infty} \widetilde{G}^{(\alpha)}_{n,\chi,q}\frac{t^n}{n!}. \tag{2.1}$$

We now consider the generalized q-Genocchi polynomials associated with χ, $\widetilde{G}^{(\alpha)}_{n,\chi,q}(x)$, are also defined by

$$\mathcal{G}^{(\alpha)}_{\chi,q}(x,t) = \frac{[2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d}e^{dt}+1}e^{xt} = \sum_{n=0}^{\infty} \widetilde{G}^{(\alpha)}_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $\chi = \chi^0, q \to 1$, above (2.1) and (2.2) will become the corresponding definitions of the Genocchi numbers G_n and polynomials $G_n(x)$. From (2.2), we note that

$$\begin{aligned}
&\frac{[2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d}e^{dt}+1}e^{xt} \\
&= \frac{[2]_{q^\alpha}}{[2]_{q^{\alpha d}}} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} \left(\frac{[2]_{q^{\alpha d}} e^{(\frac{a+x}{d})dt}}{q^{\alpha d}e^{dt}+1} \right) \\
&= \sum_{m=0}^{\infty} \left(\frac{[2]_{q^\alpha}}{[2]_{q^{\alpha d}}} d^{m-1} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} \widetilde{G}^{(\alpha)}_{m,q^d}\left(\frac{a+x}{d}\right) \right) \frac{t^m}{m!}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.1 *Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with*

$d \equiv 1 (mod 2)$. Then we have

$$\text{(1) } \widetilde{G}^{(\alpha)}_{n,\chi,q}(x) = \frac{[2]_{q^\alpha}}{[2]_{q^{\alpha d}}} d^{m-1} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} \widetilde{G}^{(\alpha)}_{m,q^d}\left(\frac{a+x}{d}\right),$$

$$\text{(2) } \widetilde{G}^{(\alpha)}_{n,\chi,q} = \frac{[2]_{q^\alpha}}{[2]_{q^{\alpha d}}} d^{m-1} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} \widetilde{G}^{(\alpha)}_{m,q^d}\left(\frac{a}{d}\right),$$

$$\text{(3) } \widetilde{G}^{(\alpha)}_{n,\chi,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \widetilde{G}^{(\alpha)}_{l,\chi,q} x^{n-l},$$

where $\widetilde{G}^{(\alpha)}_{n,q}(x)$ are the q-Genocchi polynomials with weight α (see [8]).

For $n \in \mathbb{N}$ with $n \equiv 0 (\mathrm{mod} 2)$, we have

$$\begin{aligned} &\frac{-[2]_{q^\alpha} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1} q^{\alpha n d} e^{ndt} + \frac{[2]_{q^\alpha} \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1} \\ &= \sum_{m=0}^{\infty} \left([2]_{q^\alpha} \sum_{a=0}^{nd-1} \chi(a)(-1)^a q^{\alpha a} a^m \right) \frac{t^m}{m!}. \end{aligned}$$

By (2.1), (2.2), and comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem.

Theorem 2.2 *Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, n a positive even integer, and $m \in \mathbb{N}$. Then we have*

$$\widetilde{G}^{(\alpha)}_{m+1,\chi,q} - q^{\alpha n d} \widetilde{G}^{(\alpha)}_{m+1,\chi,q}(nd) = (m+1)[2]_{q^\alpha} \sum_{a=0}^{nd-1} \chi(a)(-1)^a q^{\alpha a} a^m.$$

Next, we introduce the l-series and two variable l-series.

Definition 2.3 *For $s \in \mathbb{C}$, define two variable l-series as*

$$\widetilde{l}^{(\alpha)}_q(s, x|\chi) = [2]_{q^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) q^{\alpha m}}{(m+x)^s}.$$

By using (2.2), we easily see that

$$\begin{aligned} \mathcal{G}^{(\alpha)}_{\chi,q}(x,t) &= \frac{[2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1} e^{xt} \\ &= [2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{(a+x)t} \sum_{l=0}^{\infty} (-1)^l q^{l\alpha d} e^{dlt} \\ &= [2]_{q^\alpha} t \sum_{m=0}^{\infty} \chi(m)(-1)^m q^{\alpha m} e^{(m+x)t}. \end{aligned}$$

Then we have

$$\left(\frac{d}{dt}\right)^{k+1} \mathcal{G}_{\chi,q}^{(\alpha)}(x,t)\Big|_{t=0} = (k+1)[2]_{q^\alpha} \sum_{n=0}^{\infty} \chi(n)(-1)^n q^{\alpha n}(n+x)^k, \tag{2.3}$$

and

$$\left(\frac{d}{dt}\right)^{k+1} \left(\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}\right)\Bigg|_{t=0} = \widetilde{G}_{k+1,\chi,q}^{(\alpha)}(x), \text{ for } k \in \mathbb{N}. \tag{2.4}$$

By (2.3), (2.4), we have the following theorem.

Theorem 2.4 *For any positive integer k, we have*

$$\frac{\widetilde{G}_{k+1,\chi,q}^{(\alpha)}(x)}{k+1} = \widetilde{l}_q^{(\alpha)}(-k,x|\chi).$$

Definition 2.5 *For $s \in \mathbb{C}$, define l-series as*

$$\widetilde{l}_q^{(\alpha)}(s \mid \chi) = [2]_{q^\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) q^{\alpha m}}{m^s}.$$

By simple calculation, we have the following theorem.

Theorem 2.6 *For any positive integer k, we have*

$$\widetilde{l}_q^{(\alpha)}(-k \mid \chi) = \frac{\widetilde{G}_{k+1,\chi,q}^{(\alpha)}}{k+1}.$$

3 Witt-type formulae on $\mathbb{Z}_p$ in p-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the generalized q-Genocchi numbers $\widetilde{G}_{n,\chi,q}^{(\alpha)}$ and polynomials $G_{n,\chi,q}^{(\alpha)}(x)$ attached to χ. We assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ and $\alpha \in \mathbb{Z}$.

Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$. Let $g(y) = \chi(y)e^{(y+x)t}$. By (1.1), we derive

$$\begin{aligned} \int_X t\chi(y)e^{(y+x)t}d\mu_{-q^\alpha}(y) &= \frac{[2]_{q^\alpha} t \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d}e^{dt}+1} e^{xt} \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

By using Taylor series of $e^{(y+x)t}$ in the above equation (3.1), we obtain the Witt formula for the generalized q- Genocchi polynomials attached to χ as follows:

Theorem 3.1 *For positive integers n, we have*

$$\frac{\widetilde{G}^{(\alpha)}_{n+1,\chi,q}(x)}{n+1} = \int_X \chi(y)(y+x)^n d\mu_{-q^\alpha}(y). \tag{3.2}$$

Observe that for $x = 0$, the equation (3.2) reduces to (3.3).

Corollary 3.2 *For positive integers n, we have*

$$\frac{\widetilde{G}^{(\alpha)}_{n+1,\chi,q}}{n+1} = \int_X \chi(y)y^n d\mu_{-q^\alpha}(y). \tag{3.3}$$

By (3.2) and (3.3), we have the following theorem.

Theorem 3.3 *For positive integers n, we have*

$$\frac{\widetilde{G}^{(\alpha)}_{n+1,\chi,q}(x)}{n+1} = \sum_{l=0}^{n} \binom{n}{l} \frac{\widetilde{G}^{(\alpha)}_{l+1,\chi,q}}{l+1} x^{n-l}.$$

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