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# Operational Treatment of Three Dimensions Quantum Harmonic Oscillator

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**Abstract**

Three dimensions harmonic oscillator is one of the fundamental problems in quantum mechanics. In this research work we investigate an operational treatment of three dimensions quantum harmonic oscillator. We define lower and upper operators for three dimensions quantum harmonic oscillator and find the energy states and wave functions.

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# 1 Introduction

Oscillators is one the fundamental problems in both classical mechanics and quantum mechanics, harmonic oscillators in the former and anharmonic oscillators in latter [3, 2, 1]. There are several methods in studying oscillators in quantum mechanics. One of them is operational methods. Operational methods is useful for studying symmetries in systems. In this research work we present an operational treatment of three dimensions harmonic oscillator.

# 2 Three dimensions harmonic oscillator

The time independent Schrodinger equation for three dimensions quantum harmonic oscillator(TDQHO) in cartesian form is

$$H\psi(x_1, x_2, x_3) = E\psi(x_1, x_2, x_3) \tag{1}$$

in which

$$H = \frac{1}{2m}(p_{x_1}^2 + p_{x_2}^2 + p_{x_3}^2) + \frac{1}{2}k(x_1^2 + x_2^2 + x_3^2) \tag{2}$$

and $\frac{\hbar}{i}\frac{d}{dx_j}$ and $x_j$ are hermitian operators. In first step we investigate the energy state of TDQHO.
If we write

$$E =< \psi(x_1, x_2, x_3)|H|\psi(x_1, x_2, x_3) >= \sum_{i=1}^{3}\left(< \psi|p_{x_i}^2\psi > + \frac{1}{2} < \psi|x_i^2\psi >\right) \tag{3}$$

We know the operators $p_{x_i}$ and $x_i$ are hermitian operators and then

$$E = \sum_{i=1}^{3}\left(< p_{x_i}\psi|p_{x_i}\psi > + \frac{1}{2} < x_i\psi|x_i\psi >\right) \tag{4}$$

The $< \psi|\psi >$ is positive value and in result the energy states of TDQHO are positive $E > 0$.

## 2.1 Operational treatment

If we use the spherical coordinates, the radial equation of TDQHO is [3]

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \frac{\hbar^2}{2m}\frac{k(k+1)}{r^2}R(r) + \frac{1}{2}m\omega^2 r^2 R(r) = ER(r), \tag{5}$$

We use the change of variable $r^2 \to \rho$, then

$$-\frac{\hbar^2}{2m}4\rho\varphi''(\rho) - \frac{\hbar^2}{2m}6\varphi'(\rho) + \frac{\hbar^2}{2m}\frac{k(k+1)}{\rho}\varphi(\rho) + \frac{1}{2}m\omega^2\rho\varphi(\rho) = E\varphi(\rho) \tag{6}$$

We write the Hamiltonian of TDQHO in term of new variable as

$$H = -\frac{\hbar^2}{2m}\left(4\rho\frac{d^2}{d\rho^2} + 6\frac{d}{d\rho} - \frac{k(k+1)}{\rho}\right) + \frac{1}{2}m\omega^2\rho \tag{7}$$

If we define the operator

$$A = k - \frac{m\omega}{\hbar}\rho - 2\frac{i}{\hbar}\rho p \tag{8}$$

in which $p = \frac{\hbar}{i}\frac{d}{d\rho}$, and

$$A^\dagger = k - \frac{m\omega}{\hbar}\rho + 2\frac{i}{\hbar}p\rho \tag{9}$$

We rewrite the Hamiltonian of TDQHO in terms of $A$ and $A^\dagger$ as

$$H = \frac{\hbar^2}{2m\rho}(A^\dagger - 1)A + (k + \frac{3}{2})\hbar\omega \tag{10}$$

and we can easily find

$$[H, A] = -2\hbar\omega\left(\frac{E}{\hbar\omega} - \frac{3}{2} - \frac{m\omega}{\hbar}\rho - 2\frac{i}{\hbar}\rho p\right) \tag{11}$$

This result indicate $A$ is a lower operator. For example

$$[H, A] = -2\hbar\omega A \qquad\qquad E = (k + \frac{3}{2})\hbar\omega \tag{12}$$

Since $k \geq 0$ and from (4) we have $E > 0$, then this state is ground state for a known $k$ and we have

$$H\varphi_0(\rho) = E_0\varphi_0(\rho), \qquad\qquad E_0 = (k + \frac{3}{2})\hbar\omega. \tag{13}$$

and too

$$[H, A + 2] = -2\hbar\omega(A + 2) \qquad\qquad E = (2 + k + \frac{3}{2})\hbar\omega \tag{14}$$

As a result, $A$ is a lower operator for ground state $\varphi_0(\rho)$, $A + 2$ is a lower operator for $\varphi_1(\rho)$ and $A + 2n$ is a lower operator for $\varphi_n(\rho)$.
In the same way

$$[H, A^\dagger] = 2\hbar\omega\left(\frac{E}{\hbar\omega} - \frac{1}{2} - \frac{m\omega}{\hbar}\rho + 2\frac{i}{\hbar}p\rho\right) \tag{15}$$

This result indicate $A^\dagger$ is an upper operator. For example

$$[H, A^\dagger + 1] = 2\hbar\omega(A^\dagger + 1) \qquad\qquad E = (k + \frac{3}{2})\hbar\omega \tag{16}$$

or

$$[H, A^\dagger + 3] = 2\hbar\omega(A^\dagger + 3) \qquad\qquad E = (2 + k + \frac{3}{2})\hbar\omega \qquad (17)$$

This means $A^\dagger + 1$ is an upper operator for ground state $\varphi_0(\rho)$, $A^\dagger + 3$ is an upper operator for $\varphi_1(\rho)$ and $A^\dagger + 2n + 1$ is an upper operator for state $\varphi_n(\rho)$. By using the result (15)-(17), the energy state of TDQHO will be

$$E_n = (2n + k + \frac{3}{2})\hbar\omega \qquad\qquad n = 0, 1, 2, ... \qquad (18)$$

## 2.2 Wave functions

From (12) and (13) we have $A\varphi_0(\rho) = 0$ and

$$\varphi_0(\rho) = c_0 \rho^{\frac{k}{2}} e^{-\frac{m\omega}{2\hbar}\rho} \qquad (19)$$

where $c_0$ is normalization factor such that $\frac{1}{2}c_0^2 \int_0^\infty \varphi_0^*(\rho)\varphi_0(\rho)\rho^{\frac{1}{2}} d\rho = 1$ and

$$c_0 = \frac{\sqrt{2}}{\sqrt{(k + \frac{1}{2})!}} \left(\frac{m\omega}{\hbar}\right)^{\frac{k}{2} + \frac{3}{4}} \qquad (20)$$

The general form of radial wave function of TDQHO will write

$$\varphi_{n+1}(\rho) = c_{n+1} \prod_{i=0}^{n} (A^\dagger + 1 + 2i)\varphi_0(\rho) \qquad (21)$$

in which $c_{n+1}$ is normalization factor. For computation the normalization factor $c_n$, $n = 1, 2, 3, ...$, we need some useful relations below:

$$A = f(\rho) - A^\dagger \qquad (22)$$

in which $f(\rho) = 2 + 2k - 2\frac{m\omega}{\hbar}\rho$. We can easily find

$$[A, f(\rho)] = 4\frac{m\omega}{\hbar}\rho, \qquad\qquad [A^\dagger, f(\rho)] = -4\frac{m\omega}{\hbar}\rho \qquad (23)$$

and too

$$[A + n, \sqrt{\rho}] = -\sqrt{\rho}, \qquad\qquad n = 0, 1, 2, ... \qquad (24)$$

The last relation that we will used is

$$< \rho^n >= \frac{1}{2} c_0 c_0^\star < \varphi_0 | \rho^{\frac{1}{2}} \rho^n | \varphi_0 >= \frac{(n + k + \frac{1}{2})!}{(k + \frac{1}{2})!} (\frac{m\omega}{\hbar})^{-n} \qquad (25)$$

We start from normalization factor $c_1$

$$\varphi_1(\rho) = c_1(A^\dagger + 1)\varphi_0(\rho) \qquad (26)$$

We have

$$\frac{1}{2} < \varphi_1|\sqrt{\rho}|\varphi_1 >= \frac{1}{2}c_1^\star c_1 < \sqrt{\rho}(A^\dagger + 1)\varphi_0|(A^\dagger + 1)\varphi_0 > \tag{27}$$

By using the relations (22) and (23)

$$= \frac{1}{2}c_1^\star c_1 < (1 + f(\rho) - A^\dagger)\sqrt{\rho}(1 + f(\rho) - A)\varphi_0|\varphi_0 >$$

and

$$= \frac{1}{2}c_1^\star c_1 < (1 + f(\rho))^2\sqrt{\rho}\varphi_0|\varphi_0 >= 1$$

and by (25)

$$c_1 = \frac{1}{2\sqrt{k + \frac{3}{2}}}$$

In the same way

$$\varphi_2(\rho) = c_2(A^\dagger + 3)(A^\dagger + 1)\varphi_0(\rho) \tag{28}$$

and

$$\frac{1}{2} < \varphi_2|\sqrt{\rho}|\varphi_2 >= \frac{1}{2}c_2^\star c_2 < \sqrt{\rho}(A^\dagger + 3)(A^\dagger + 1)\varphi_0|(A^\dagger + 3)(A^\dagger + 1)\varphi_0 >$$

$$= \frac{1}{2}c_2^\star c_2 < ((1 + f(\rho))(3 + f(\rho)) - 4\frac{m\omega}{\hbar}\rho)^2\sqrt{\rho}\varphi_0|\varphi_0 >$$

$$c_2^\star c_2 32(k + \frac{3}{2})(k + \frac{5}{2}) = 1$$

and

$$c_2 = \frac{1}{2\sqrt{1}\sqrt{(k + \frac{3}{2})}}\frac{1}{2\sqrt{2}\sqrt{(k + \frac{5}{2})}} \tag{29}$$

For an arbitrary radial wave function

$$\varphi_{n+1}(\rho) = c_{n+1}\prod_{i=0}^{n}(A^\dagger + 1 + 2i)\varphi_0(\rho) \tag{30}$$

in which

$$c_{n+1} = \frac{1}{2\sqrt{1}\sqrt{k + \frac{3}{2}}}\frac{1}{2\sqrt{2}\sqrt{k + \frac{5}{2}}}...\frac{1}{2\sqrt{n}\sqrt{k + \frac{2n+1}{2}}}$$

$$= \frac{\sqrt{(k + \frac{1}{2})!}}{2^n\sqrt{n!}\sqrt{(k + \frac{2n+1}{2})!}} \tag{31}$$

### 2.3 Orthogonality

We can verify the orthogonality of $\varphi_n(\rho)$ and $\varphi_m(\rho)$, for $m > n$

$$< \varphi_n(\rho)|\sqrt{\rho}|\varphi_m(\rho) >= c_m < \sqrt{\rho}\varphi_n(\rho)| \prod_{i=0}^{m-1}(A^\dagger + 1 + 2i)\varphi_0(\rho) >$$

$$= c_m < (A+3+2n)...(A-1+2m)(A+1)(A+3)...(A+1+2n)\sqrt{\rho}\varphi_n(\rho)|\varphi_0(\rho) >$$

by using (24)

$$= c_m < (A+3+2n)...(A-1+2m)A(A+1)(A+3)...\sqrt{\rho}(A+2n)\varphi_n(\rho)|\varphi_0(\rho) >$$

$A + 2n$ is a lower operator for $\varphi_n(\rho)$

$$= c_m c'^{\star}_{n-1} < (A+3+2n)...(A-1+2m)(A+1)(A+3)...(A-1+2n)\sqrt{\rho}\varphi_{n-1}(\rho)|\varphi_0(\rho) >$$

after several time

$$= c_m c'^{\star}_{n-1}...c'^{\star}_0 < (A+3+2n)...(A-1+2m)(A+1)\sqrt{\rho}\varphi_0(\rho)|\varphi_0(\rho) >$$

by using of (24)

$$= c_m c'^{\star}_{n-1}...c'^{\star}_0 < (A+3+2n)...(A-1+2m)\sqrt{\rho}A\varphi_0(\rho)|\varphi_0(\rho) >= 0 \qquad (32)$$

# 3 Conclusion

In this research work we present a complete treatment of three dimensions quantum harmonic oscillator.

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