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Spatial Distribution of the Magnetic Field Generated by a Circular Arc Current

Dekui Zhao and Lili Gao

Sichuan University of Science and Engineering,
Zigong, Sichuan 643000, China
dkzllg@126.com

Abstract

The magnetic potential generated by a circular arc current is calculated. Taking gradient operation gives a closed-form expression for the spatial distribution of the magnetic induction in terms of elliptic integrals, which is useful for analysis and computation. The characteristics of the field distribution are demonstrated. Some special cases are discussed, illustrated and numerically computed, including the field due to a circular current loop.

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1. Introduction

Analytical calculation of the magnetic field is still of interests to the readership[1-3]. Compared to the magnetic field of a circular current [4], the spatial distribution of the magnetic field generated by a circular arc current is more general and useful, which is beyond a textbook problem. The prime motivation of this paper is to obtain a closed form expression of the magnetic field produced by a circular arc current. That expression is convenient for both theoretic analysis and numerical computation. The properties of the field distribution are discussed and illustrated. Moreover, some special cases are presented.

2. Magnetic potential

A circular arc current I with radius a and central angle β_0 is depicted in Fig. 1. With cylindrical coordinates the position of point P is denoted as (ρ, φ, z).

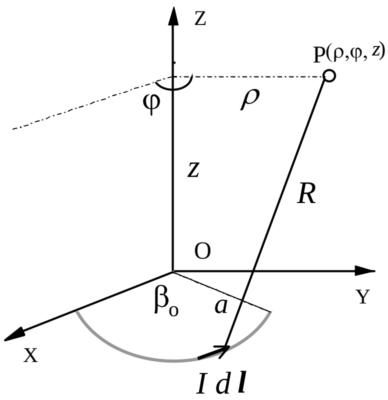

Fig.1 A circular arc current

On the arc choose a current element $Id\boldsymbol{l} = \boldsymbol{e}_\beta\, Iad\beta$ with angular coordinate β. It produces magnetic potential $d\mathbf{A}$ at P :

$$d\mathbf{A} = \frac{\mu_0 I d\boldsymbol{l}}{4\pi R} = \boldsymbol{e}_\rho \frac{\mu_0 I a \sin(\beta - \varphi) d\beta}{4\pi R} + \boldsymbol{e}_\varphi \frac{\mu_0 I a \cos(\beta - \varphi) d\beta}{4\pi R} \tag{1}$$

where R is the distance from P to that current element

$$R = \sqrt{a^2 + \rho^2 + z^2 - 2a\rho\cos(\beta - \varphi)} \tag{2}$$

Introducing transformation $\Phi = \frac{1}{2}(\pi + \beta - \varphi)$, the φ –component of the total magnetic potential at P is given by

$$\begin{aligned} A_\varphi &= -\frac{\mu_0 I a}{2\pi} \int_{\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi+\beta_0)} \frac{\cos 2\Phi d\Phi}{\sqrt{a^2 + \rho^2 + z^2 + 2a\rho\cos 2\Phi}} \\ &= \frac{\mu_0 I}{4\pi\rho\sqrt{(a+\rho)^2 + z^2}} \left[(a^2 + \rho^2 + z^2) \int_{\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi+\beta_0)} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} - \left((a+\rho)^2 + z^2\right) \int_{\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi+\beta_0)} \sqrt{1 - k^2 \sin^2 \Phi}\, d\Phi \right] \\ &= \frac{\mu_0 I}{4\pi\rho\sqrt{(a+\rho)^2 + z^2}} \left[(a^2 + \rho^2 + z^2) \left[F\left(\frac{\pi - \varphi + \beta_0}{2}, k\right) - F\left(\frac{\pi - \varphi}{2}, k\right) \right] - \left((a+\rho)^2 + z^2\right) \left[E\left(\frac{\pi - \varphi + \beta_0}{2}, k\right) - E\left(\frac{\pi - \varphi}{2}, k\right) \right] \right] \end{aligned} \tag{3}$$

where $F(\theta,k)$ and $E(\theta,k)$ are Legendre elliptic integrals of the first and second kinds

$$F(\theta,k)=\int_0^{\theta}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}};\quad E(\theta,k)=\int_0^{\theta}\sqrt{1-k^2\sin^2\alpha}\,d\alpha \tag{4}$$

And the modulus *k* is

$$k=2\sqrt{\frac{a\rho}{(a+\rho)^2+z^2}}\le 1 \tag{5}$$

The ρ –component of the total magnetic potential at *P* is calculated as

$$A_\rho=\frac{\mu_0 Ia}{4\pi}\int_0^{\beta_0}\frac{\sin(\varphi-\beta)d\beta}{\sqrt{a^2+\rho^2+z^2-2a\rho\cos(\varphi-\beta)}}=-\frac{\mu_0 I}{4\pi\rho}\left(\sqrt{a^2+\rho^2+z^2-2a\rho\cos(\varphi-\beta_0)}-\sqrt{a^2+\rho^2+z^2-2a\rho\cos\varphi}\right) \tag{6}$$

Apparently, the total magnetic potential at *P* has no *z*-component. That is

$$A_z=0 \tag{7}$$

3. Magnetic induction

Denote the unit vector on the direction of ρ , φ, and z as $\mathbf{e}_\rho$,$\mathbf{e}_\varphi$ and $\mathbf{e}_z$, respectively. The spatial distribution of the magnetic field due to the arc current could be determined through calculating the curl of ***A***[5]

$$\boldsymbol{B}=\nabla\times\boldsymbol{A}=-\boldsymbol{e}_\rho\frac{\partial A_\varphi}{\partial z}+\boldsymbol{e}_\varphi\frac{\partial A_\rho}{\partial z}+\boldsymbol{e}_z\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi)-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right) \tag{8}$$

Taking the derivative of $F(\theta,k)$ and $E(\theta,k)$ with respect to k in Eqs.(4), we arrive at

$$\frac{dF(\theta,k)}{dk}=\frac{E(\theta,k)}{k(1-k^2)}-\frac{F(\theta,k)}{k}-\frac{k\sin 2\theta}{2(1-k^2)\sqrt{1-k^2\sin^2\theta}} \tag{9}$$

$$\frac{dE(\theta,k)}{dk}=\frac{E(\theta,k)-F(\theta,k)}{k} \tag{10}$$

Using the two above results in Eq.(8) we obtain the components of the magnetic induction as, after simplifying

$$B_\rho = \frac{\mu_0 I k z}{8\pi\rho\sqrt{a\rho}}\left(-\Delta F + \Psi\,\Delta E - \frac{1}{2}k^2\Psi\Delta\Omega\right);\quad B_\varphi = -\frac{\mu_0 I k z}{8\pi\,\rho\sqrt{a\rho}}\Delta\omega\ ;$$

$$B_z = \frac{\mu_0 I k}{8\pi\sqrt{a\rho}}\left(\Delta F + \psi\,\Delta E - \frac{1}{2}k^2\psi\Delta\Omega\right) \tag{11}$$

where

$$\Delta F = F\left(\frac{\pi-\varphi+\beta_0}{2},k\right) - F\left(\frac{\pi-\varphi}{2},k\right);\quad \Delta E = E\left(\frac{\pi-\varphi+\beta_0}{2},k\right) - E\left(\frac{\pi-\varphi}{2},k\right);$$

$$\Psi = \frac{a^2+\rho^2+z^2}{(a-\rho)^2+z^2};\qquad \psi = \frac{a^2-\rho^2-z^2}{(a-\rho)^2+z^2};$$

$$\Delta\Omega = \frac{\sin(\varphi-\beta_0)}{\sqrt{1-\frac{k^2}{2}\left(1+\cos(\varphi-\beta_0)\right)}} - \frac{\sin\varphi}{\sqrt{1-\frac{k^2}{2}\left(1+\cos\varphi\right)}};$$

$$\Delta\omega = \frac{1}{\sqrt{1-\frac{k^2}{2}\left(1+\cos(\varphi-\beta_0)\right)}} - \frac{1}{\sqrt{1-\frac{k^2}{2}\left(1+\cos\varphi\right)}} \tag{12}$$

Obviously, the magnitude of the magnetic induction ***B*** at *P* is

$$B = \sqrt{B_\rho^2 + B_\varphi^2 + B_z^2} \tag{13}$$

4. Advantages for analyzing, computing and plotting

In comparison with the numerical result by direct computations based on Biot-Savart law the group of Eqs. (11) has two advantages. First, it is a general analytic solution with a closed-form expression for magnetic field of an arc current. It certainly satisfies $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{B}=0$ and $\nabla^2\mathbf{A}=0$ in all space except on the arc. To verify it not only helps our readership to gasp the main properties of magnetic field but also to arouse the interest in applied math. Furthermore, since it has been easy to call the various Legendre elliptic integrals in some advanced software such as *Mathematica*[6] , the group of Eqs. (11) is more convenient and intuitive to make a program for computation of numerical values and plot of graphical solutions on a PC. Taking the field of a half circular current ($\beta_0 = \pi$) as an example, we compute the magnitude distribution on the column surface $\rho = 2a$ by using *Mathematica*. Introducing the dimensionless magnitude $b = \frac{8\pi a}{\mu_0 I} B$, Fig.2 shows the function of b versus z/a and φ.

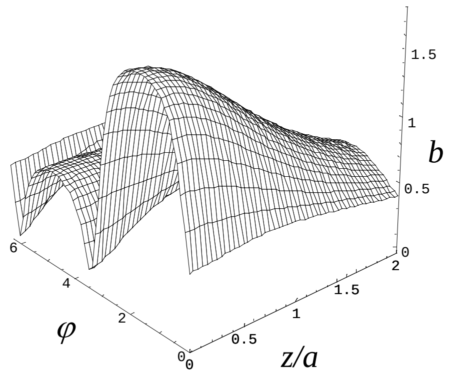

Fig. 2 b versus z/a and φ under the condition of $\beta_0 = \pi$ and $\rho = 2a$

5. Special cases

5.1. Along the central axis

The distribution of magnetic field along the axis through the center and perpendicular to the plane of the arc may be get by letting $\rho \to 0$ in Eqs. (11)-(13). Employing the expansions of Legendre elliptic integrals and de l'Hôpital law, along the central axis we have

$$B_\rho = mz[\sin\varphi + \sin(\beta_0 - \varphi)]; \quad B_\varphi = mz[\cos\varphi - \cos(\beta_0 - \varphi)]; \quad B_z = ma\beta_0 \tag{14}$$

$$B = m\sqrt{a^2\beta_0^2 + 2z^2(1-\cos\beta_0)} \tag{15}$$

where

$$m = \frac{\mu_0 I a}{4\pi(a^2 + z^2)^{3/2}} \tag{16}$$

Eqs.(14) shows that $\varphi = \beta_0/2$ causes $B_\varphi = 0$. It implies that along the central axis the direction of ***B*** is in the plane determined by the axis and the midpoint of the arc.

5.2. In the plane of the arc

In the plane of the arc we have $z = 0$. Thus Eqs. (11) reduce to

$$B_\rho = 0; \qquad B_\varphi = 0; \qquad B_z = \frac{\mu_0 I}{4\pi(a+\rho)}\left(\Delta F + \frac{a+\rho}{a-\rho}(\Delta E - \frac{1}{2}k_0^2 \Delta\Omega)\right) \tag{17}$$

where the modulus k_0 is the ratio of the geometric mean to the arithmetic mean for *a* and ρ

$$k_0 = \frac{2\sqrt{a\rho}}{a+\rho} \tag{18}$$

The curve of b_z as a function of φ under the condition of $\beta_0 = \pi$ and $z = 0$ is plotted by *Mathematica* in Fig.3. There are two zero points of B_z. Through numerical computation we know they are exactly at $\varphi_1 =$ 3.4051260960033936 and $\varphi_2 = 6.01965186476598615$.

It will be seen from Eqs.(17) that in finite space all points of $B = 0$ are in the plane of the arc current. Using the command ContourPlot we can ask *Mathematica* to

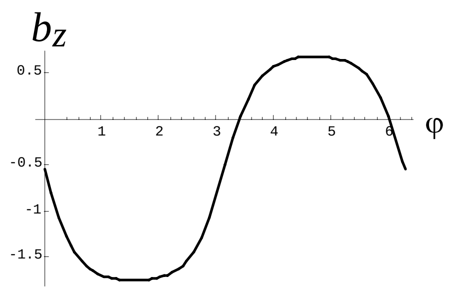

Fig. 3 b_z versus φ under the condition of $\beta_0 = \pi$ and $z = 0$

plot the curve of $B = 0$ in the plane of a half circular current. The graphic is shown in Fig.4.

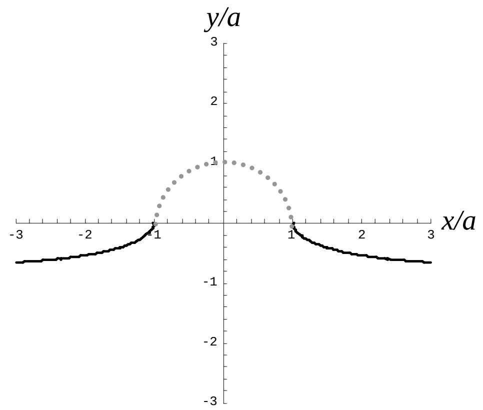

Fig. 4 The curve of $B = 0$ in the plane of a half circular current

5.3. Circular current loop

Consider the condition for the circular current loop $\beta_0 = 2\pi$. The periodicity of the integrand in Eqs.(4) leads to[7]

$$F(\pi+\theta,k) - F(\theta,k) = 2K(k)\,; \qquad E(\pi+\theta,k) - E(\theta,k) = 2E(k) \tag{19}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively. Applying the two above results to Eqs.(11) yields

$$B_\rho = \frac{\mu_0 I\, kz}{4\pi\rho\sqrt{a\rho}}\left[-K(k) + \psi\, E(k)\right]; \qquad B_\varphi = 0\,; \qquad B_z = \frac{\mu_0 I k}{4\pi\sqrt{a\rho}}\left[K(k) + \psi\, E(k)\right] \tag{20}$$

Noticing $K(0) = E(0) = \dfrac{\pi}{2}$, along the central axis we get

$$B = B_z = 2\pi a m = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \tag{21}$$

And in the plane of the arc we have

$$B = B_z = \frac{\mu_0 I}{2\pi}\left[\frac{K(k_0)}{a+\rho} + \frac{E(k_0)}{a-\rho}\right] \tag{22}$$

Especially, at the central point

$$B = B_z = \frac{\mu_0 I}{2a} \tag{23}$$

We have been acquainted with Eqs. (21) and (23) already[8].

6. Conclusion

The spatial distribution of the magnetic field established by an arc current can be accurately achieved with the elliptic integrals. That closed-form expression is convenient for theoretic analysis and intuitive for numerical computation and plot on a PC. The general result contains some typical special cases, including the distribution of the field set up by a circular current loop.

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