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Spinor Field Equation of Arbitrary Spin in Robertson-Walker Space-time: Solution

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Abstract

The study of the field equation of arbitrary spin in Robertson-Walker space-time, previously separated by variable separation, is completed. The integration of the separated radial equations is performed in an unified way with respect to the curvature parameter. Through a sequence of transformations on the variable and of the radial function, the radial equation is reported to the Heun's differential equation. The solution of the Heun's equation however does fall into the class of known functions such as the hypergeometric, the polynomial, the polynomial-like function, only exceptionally. Moreover the Heun's differential operator admits of a factorization, a property that would simplify the integration, only for special values of the parameters.

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1 Introduction

The solution of the field equation of arbitrary spin in space-times of physical relevance is a subject of attraction. It is of intereset both from a mathematical point of view as well as for physical applications and cosmological implications. It is also a necessary condition for further developments of the theory. In particular the normal modes solutions are needed for the quantization of the field. The field equations in curved space-time have been widely considered

[2, 4, 8, 15]. Many methods of solution are indebted to the pioneeer paper by Chandrasekhar [3] who succeeded in separating the Dirac equation in Kerr metric. In the Schwarzschild geometry the field equation can be separated for arbitrary value of the spin [25]. The integration of the equation has been extended also to time dependent metric (e.g., [9, 10] and References therein). Recently [27], the field equation of arbitrary spin has been separated in the Robertson-Walker (RW) space-time, the angular equation integrated as well as the flat case of the separated radial equation. Also the time dependence, that results in a pair of coupled differential equations, has been integrated for some examples of cosmological time evolution of physical interest. The mentioned results have been obtained as a generalization of results relative to specific values of the spin, [19, 20, 21, 22, 23, 24, 26]. In the general treatment of Ref. [27] the problem was left open of the solution of the general radial equation in the open and closed space-time case. Recently, for the spin $s = 1$ the problem has been studied [28] and the radial equation has been reduced to the Heun's equation, a recently widely reconsidered equation [11, 16].

The object of the present paper is of generalizing the results of Ref. [28] in RW space-time to the case of arbitrary spin field. To that end we first recollect the results previously obtained [27]. Then the separated radial equation are treated in an unified manner for both the open and closed space-time case. After a sequence of modifications of the independent variable and of the radial function, the radial equation is reduced to a differential equation of Heun's type [7, 11, 16]. On the base of the Fuchsian character of the equation, the asymptotic behaviours of its solutions are explicited. They are coherent with previous results in flat space-time case.

As for the spin 1 field case the configuration of the parameters of Heun's equation are such that the present case does not generically fall into known situations. In almost all cases of the parametric configurations, the solution cannot be expressed in terms of hypergeometric function [11]. In general it has not polynomial [5, 6] or a polynomial like [16] form, nor the Heun's differential operator can be factorized [17]. The Heun's equation can be however formally integrated [5, 6]. One can also proceed by the standard power series integration of the Fuchsian equations [14, 17]. Both last methods however do not seem to furnish sufficient analytical informations on the solutions as needed for further developments of the theory. This is the case, as mentioned, for the determination of the normal modes of eq. (1), in view of a quantization of the field. As far as the author knows, this problem has been solved, apart the Dirac equation [13], only for the spin 1 field in the flat space-time case [24].

2 Preliminary Assumptions.

The field equation for spin s assumes a simplified form in a conformally flat space-time (see, e.g., [8, 15] and references therein). In the spinor formulation it reads:

$$\begin{aligned}\nabla^{A}_{\dot{X}}\,\phi_{AA_1\ldots A_n} + \mu_\star \chi_{A_1A_2\ldots A_n\dot{X}} &= 0\\ \nabla^{\dot{Z}}_{A}\,\chi_{A_1A_2\ldots A_n\dot{Z}} - \mu_\star \phi_{AA_1A_2\ldots A_n} &= 0,\end{aligned}\tag{1}$$

where $n = 0, 1, 2, ..,$ and where $\sqrt{2}\mu_\star = im_0,\ m_0$ the mass of the particles of the field; s is the value of the spin $s = (n+1)/2$ and the spinor fields are assumed to be symmetric in the undotted indexes: $\phi_{AA_1\ldots A_n} = \phi_{(AA_1\ldots A_n)}$ and $\chi_{A_1A_2\ldots A_n\dot{X}} = \chi_{(A_1A_2\ldots A_n)\dot{X}}$. By setting

$$\begin{aligned}\phi_h &\equiv \phi_{AA_1A_2\ldots A_n} \;\Leftrightarrow\; A + A_1 + \ldots + A_n = h, \quad h = 0, 1, 2, \ldots, n+1\\ \chi_{j\dot{X}} &\equiv \chi_{A_1A_2\ldots A_n\dot{X}} \;\Leftrightarrow\; A_1 + A_2 + \ldots A_n = j, \quad j = 0, 1, \ldots, n\end{aligned}\tag{2}$$

The equation (1) can be separated in the Robertson-Walker (RW) metric [18] of line element

$$ds^2 = dt^2 - R(t)^2\Big[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\Big], \quad a = 0, \pm 1 \tag{3}$$

This is possible through variable separation by setting

$$\begin{aligned}\phi_j(t, r, \theta, \varphi) &= \alpha(t)\phi_j(r)S_j(\theta)\exp(im\varphi), \quad j = 0, 1, \ldots, n+1\\ \chi_{h\dot{0}}(t, r, \theta, \varphi) &= A(t)\phi_{h+1}(r)S_{h+1}(\theta)\exp(im\varphi),\\ \chi_{h\dot{1}}(t, r, \theta, \varphi) &= -A(t)\phi_h(r)S_h(\theta)\exp(im\varphi), \quad h = 0, 1, \ldots, n\end{aligned}\tag{4}$$

where $m = 0, \pm 1, \pm 2, \pm 3, \ldots$.
The resulting separated angular equation has been integrated. For $m \geq 0$, the solutions of the angular equation are

$$S_j = (1 - \cos\theta)^{\frac{m+s-j}{2}}(\cos\theta + 1)^{\frac{m-s+j}{2}} P^{(m+j-s, m-j+s)}_{l-m}(\cos\theta), \quad m \geq 0 \tag{5}$$

$P^{(\alpha,\beta)}_n$ the Jacobi polynomials [1]. For $m < 0$ the S_j's are obtained from (5) by the substitutions $m \to |m|$, $\cos\theta \to -\cos\theta$. In any case it holds $l = |m|, |m| + 1, |m| + 2, \ldots$ [27].
The separated time dependence consists in a pair of coupled equations in $\alpha(t),\ A(t)$:

$$\begin{aligned}\dot{\alpha}R + (s+1)\,\alpha\dot{R} - im_0\,AR &= -ik\alpha\\ \dot{A}R + (2-s)\,A\,\dot{R} - im_o\alpha\,R &= ikA\end{aligned}\tag{6}$$

They can be integrated once the explicit cosmological time evolution $R(t)$ has been assigned. Examples of physical interest have been discussed in [21, 27]. As to the separated radial equation, one is left with

$$r(1-ar^2)\phi_j'' + [2s+2-(2s+3)ar^2]\phi_j' + \Big\{r\Big[k^2 - a(j+1)(2s+1-j)\Big] \\ +2ik(s-j)\sqrt{1-ar^2} + \tfrac{2s-\lambda^2}{r}\Big\}\phi_j = 0 \qquad (7)$$

$$\lambda^2 = l(l+1) - s(s-1), \quad a = 0, \pm 1 \qquad (8)$$

The equation (7) results invariant under the substitution $j \to n+1-j$ plus complex conjugation. Therefore one has the relations $\phi_{n+1-j} \cong \phi_j^\star$. For $a = 0$ the equation (7) can be reduced to a confluent hypergeometric equation so that the solution is finally expressed [27] in terms of Kummer function [1]. A possible solution is

$$\phi_j(r) \;\cong\; r^{l-s}\exp(ikr)M(s+l+1-j; 2l+2; -2ikr) \quad (a=0) \qquad (9)$$

(There is indeed a second independent solution that, on account of the fact that $2l+2$ is a positive integer, is related to the "logaritmic" solution of the confluent hypergeometric equation [1]). The solution of eq. (7) for $a = \pm 1$ has not yet been given (as far as the author knows) except for $s = 1$ (e.g., [28]. For $s = 1/2$ see e.g. [19]).

3 Reduction to Heun's equation.

The equation (7) can be reduced to the Heun's equation in an unified manner for $a = \pm 1$ through the following steps:

$$r = \frac{1}{\sqrt{-a}}\sinh(\sqrt{-a}x) \qquad (10)$$

$$t = \frac{1}{\sqrt{-a}}\tanh(\sqrt{-a}\frac{x}{2}) \qquad (11)$$

$$\phi_j = t^\alpha Z_j(t), \quad \alpha = l-s, -l-s-1 \qquad (12)$$

$$t = \tau\sqrt{-a} \qquad (13)$$

$$Z_j = (\tau-1)^{1+ik\sqrt{-a}+j}(\tau+1)^{1-ik\sqrt{-a}+j}V_j(\tau) \qquad (14)$$

After the substitutions (10), (11), the equation (7) becomes

$$\frac{d^2\phi_j}{dt^2} + 2\frac{[1+s(1-t^2)]}{t(1+at^2)}\frac{d\phi_j}{dt} + \Big\{\frac{4[k^2-a(j+1)(2s+1-j)]}{(1+at^2)^2} + \\ + \frac{4ik(s-j)(1-at^2)}{t(1+at^2)^2} + \frac{2s-\lambda^2}{t^2}\Big\}\phi_j = 0, \quad a = \pm 1 \qquad (15)$$

By applying the transformations (12), (13) to eq. (15) one obtains then

$$\frac{d^2 Z_j}{d\tau^2} + \frac{2\tau^2(\alpha - s) - 2(\alpha + s + 1)}{\tau(\tau^2 - 1)}\frac{dZ_j}{d\tau} - \Big\{\frac{2\alpha(2s+1)}{\tau^2 - 1} + + \frac{4a[k^2 - a(j+1)(2s+1-j)]}{(\tau^2-1)^2} - \frac{4ik(s-j)\sqrt{-a}(1+\tau^2)}{\tau(\tau^2-1)^2}\Big\}Z_j = 0 \quad (16)$$

Finally we use position (14) into eq. (16). (Notice that in eq. (14) there is a particular choice of the characteristic exponents for equation (16) in $\tau = \pm 1$). The resulting equation for V_j is

$$\begin{aligned} \frac{d^2V_j}{d\tau^2} &+ \Big[\frac{2(\alpha+s+1)}{\tau} + \frac{2j - 2s + 1 + 2ik\sqrt{-a}}{\tau - 1} + \\ &+ \frac{2j - 2s + 1 - 2ik\sqrt{-a}}{\tau + 1}\Big]\frac{dV_j}{d\tau} + \\ &+ 2(1 + j + \alpha)\frac{2ik\sqrt{-a} + \tau(2j - 2s + 1)}{\tau(\tau-1)(\tau+1)}V_j = 0 \end{aligned} \quad (17)$$

The last equation is a Heun's equation [11,16] that with the standard identification of the parameters reads

$$\frac{d^2V_j}{d\tau^2} + \Big(\frac{\gamma}{\tau} + \frac{\delta}{\tau - 1} + \frac{\epsilon}{\tau + 1}\Big)\frac{dV_j}{d\tau} + \frac{\bar\alpha\bar\beta\tau - q}{\tau(\tau-1)(\tau+1)}V_j = 0 \quad (18)$$

$$\gamma = 2(\alpha + s + 1) \qquad q = -4ik\sqrt{-a}(1 + \alpha + j) \quad (19)$$

$$\delta = 2j - 2s + 1 + 2ik\sqrt{-a} \qquad \bar\alpha = 2j + 2\alpha + 2 \quad (20)$$

$$\epsilon = 2j - 2s + 1 - 2ik\sqrt{-a} \qquad \bar\beta = 2j - 2s + 1 \quad (21)$$

The parameters are linked, in general, by the relation $\epsilon = \bar\alpha + \bar\beta - \delta - \gamma + 1$ that is easily checked to be satisfied in the present case. The characteristic exponents α_i, $i = 0, 1, -1, \infty$ of the indicial equation of eq. (18) relative to the regular singularities in $0, \pm 1, \infty$ are

$$\alpha_0 = 0, \quad 1 - \gamma \equiv -2\alpha - 2s - 1 \quad (22)$$

$$\alpha_1 = 0, \quad 1 - \delta \equiv 2s - 2j - 2ik\sqrt{-a} \quad (23)$$

$$\alpha_{-1} = 0, \quad 1 - \epsilon \equiv 2s - 2j + 2ik\sqrt{-a} \quad (24)$$

$$\alpha_\infty = \bar\alpha, \bar\beta, \quad \bar\alpha \equiv 2(j + 1 + \alpha), \ \bar\beta \equiv 2j - 2s + 1 \quad (25)$$

From the general properties of the Fuchsian equation there are two linear independent solutions of eq. (17) in the neighbourhood of any singular point. For $\tau = 0$ two local solutions are [11, 12]

$$V_j^{(1)} = Hl(-1, q; \bar\alpha, \bar\beta, \gamma, \delta; \tau) \quad (26)$$

$$\begin{aligned} V_j^{(2)} &= \tau^{1-\gamma} Hl(-1, q - (\gamma - 1)(\bar\alpha + \bar\beta - \gamma - 2\delta + 1); \\ &\quad ; \beta - \gamma + 1, \alpha - \gamma + 1, 2 - \gamma, \delta; \tau) \end{aligned} \quad (27)$$

where the "Heun local" solution is of the form $Hl = \sum_n c_n \tau^n$ with $c_0 = 1$. The 192 local solutions of Heun's equation have been recently catalogued in [12]. The local radial solutions corresponding to (26) (27) are then

$$\phi_j^{(h)}(r) = \tau^\alpha (\tau-1)^{1+j+ik\sqrt{-a}} (\tau+1)^{1+j-ik\sqrt{-a}}\, V_j^{(h)}(\tau), \quad h = 1,2 \quad (28)$$
$$\alpha = l-s; -l-s-1, \qquad \tau = -\frac{1}{a}\tanh\left(\frac{1}{2}\sinh^{-1}(r\sqrt{-a})\right)$$

For $r \to 0$ one has $\tau \simeq \frac{1}{\sqrt{-a}}\frac{r}{2}$. Therefore one finds

$$\phi_j^{(1)} \overset{r\to 0}{\sim} r^\alpha, \qquad a = 0, \pm 1 \quad (29)$$
$$\phi_j^{(2)} \overset{r\to 0}{\sim} r^{-\alpha-2s-1}, \qquad \alpha = l-s, -l-s-1 \quad (30)$$

The last relations hold also for $a = 0$ as it follows from eq. (7) [27]. Note that the behaviour of $\phi_j^{(1)}$ relative to the two value of α are exchanged when passing to $\phi_j^{(2)}$.

The local radial solutions in the neighbourhood of $\tau = 1$, $(a = -1)$ are given by:

$$\phi_j^{(h)}(r) = t^\alpha (t+1)^{1+j+ik}(t-1)^{1+j-ik} V_j^{(h)}(t), \quad h = 1,2 \quad (a=-1) \quad (31)$$
$$V_j^{(1)} = Hl(2, -q+\bar\alpha\bar\beta; \bar\alpha, \bar\beta, \delta, \gamma; 1-t) \quad (32)$$
$$V_j^{(2)} = (t-1)^{2s-2j-2ik} Hl(2, -q+\gamma(1-\delta)+(\bar\beta-\delta+1)(\bar\alpha-\delta+1); \; ;\bar\beta-\delta+1, \bar\alpha-\delta+1, 2-\delta; 1-t) \quad (33)$$

$\tau = t = -\tanh\left[\frac{1}{2}\sinh^{-1} r\right]$. Note that $t \to 1$ for $r \to \infty$. For $r \to \infty$, $\sinh^{-1} r \sim \log 2r$, $t-1 \sim -\frac{1}{r}$ and hence one obtains

$$\phi_j^{(1)} \overset{r\to\infty}{\sim} \frac{e^{-ik\log r}}{r^{j+1}} \qquad j < s \quad (34)$$
$$\phi_j^{(1)} \overset{r\to\infty}{\sim} \frac{e^{ik\log r}}{r^{1+2s-j}} \qquad j > s \quad (35)$$
$$\phi_j^{(1)} \overset{r\to\infty}{\sim} \frac{\cos k\log r}{r^{s+1}} \qquad j = s \quad (36)$$

The behaviour (35) follows from (34) and the mentioned relation $\phi_{2s-j} \cong \phi_j^\star$. The behaviour (36) follows by taking a real combination of (34), (35). [To be precise the behaviour (35) is a valid behaviour also for $j < s$ as well as (34) is a valid behaviour also or $j > s$. This can be seen by considering all the characteristic exponents of eq. (16) for $t = \tau = 1$, $a = -1$.] Of course $s = j$ is possible only for integer spin, that is for bosons. Similarly

$$\phi_j^{(2)} \overset{r\to\infty}{\sim} \frac{e^{ik\log r}}{r^{1-j+2s}} \qquad j < s \quad (37)$$

$$\phi_j^{(2)} \overset{r\to\infty}{\sim} \frac{e^{-ik\log r}}{r^{1+j}} \qquad j > s \tag{38}$$

$$\phi_j^{(2)} \overset{r\to\infty}{\sim} \frac{\cos k\log r}{r^{1+s}} \qquad j = s \tag{39}$$

so that the behaviour of $\phi_j^{(2)}$ for $j < s,\ j > s$ is that of $\phi_j^{(1)}$ for $j > s,\ j < s$ respectively.

We now consider the behaviour of ϕ_j in $r = 1$ when $a = 1$. If $r \to 1^-$, then $t \to 1,\ \tau \to -i$ and $\sin^{-1} r \simeq \frac{\pi}{2} + \sqrt{2(1-r)},\ t-1 \simeq \sqrt{2(1-r)}$. Since $\tau = -i$ is a regular point of, e.g., eq. (17), the corresponding characteristic exponents are trivial: $\alpha_{-i} = 0, 1$. By choosing $\alpha_{-i} = 1$, one has therefore

$$\phi_j \overset{r\to 1^-}{\sim} \tau + i \sim t - 1 \sim (1-r)^{\frac{1}{2}} \tag{40}$$

The correctness of the result can be directly checked from eq. (7) where the characteristic non trivial exponent in $r = 1$ results to be 1/2.

4 Remarks and Comments

In the previous Sections the field equation of arbitrary spin has been integrated in the Robertson-Walker space-time. It is usefull however to remark that the Heun's equation to which the radial equation has been reduced, admits of known solution only exceptionally. To see this suppose first $q = 0$ in eqs. (18)-(21). Then $k = 0$ or $\bar{\alpha}\bar{\beta} = 0$ or both. If $k = 0$ and $\bar{\alpha}\bar{\beta} \neq 0$, the equation can be reported to an hypergeometric equation. If $k = 0$ and $\bar{\alpha}\bar{\beta} = 0$ the Heun's equation can be easily integrated.

Let now $q \neq 0$, in (18), so that $p \equiv q/(\bar{\alpha}\beta) \neq 0$. Then, by Theorem 3.7a) of Ref. [11], the Heun's equation is not reducible to an hypergeometric equation with solution of the type $Hl(\tau) =_2 F_1(R(\tau)),\ R$ a rational function.

Explicit solutions could be obtained by providing a factorization of the Heun's differential operator (e.g. [7, 17]). This is possible however if the parameters satisfy at least one of the special relations listed in [17]. If these relations are tested in our case (18)-(21), one finds that some of them are not at all satisfied while the others are satisfied by special values or relations among the labels j, l, s, k.

One could also ask whether the present Heun's equation admits of polynomials or "polynomial like" solutions. For polynomial solutions it must be, as pointed out in [6], $q = 0$ or $q = (\epsilon - \delta)(\gamma - 1)$. The last condition is satisfied only for $k = 0$ or, if $k \neq 0$, if $j = \alpha + 2s$. Similarly the Heun's equation admits of "polynomial-like" solutions only if the specific relations listed in [16] are verified for j, l, s, k.

According to the above discussion there follows that solutions of the radial equation in the open and closed RW metric is, in general, not simple. If one has to consider “packets” of particular solutions, that is superposition of solutions relative to different k, l, m, then the problem may become very cumbersome.

The Heun’s equation can of course be solved both formally [5, 6] both by series in the line of the integration of Fuchsian equations by power series [14, 16] . However, besides the properties of the solutions given by these methods, further analytical informations of Heun’s solutions would be appreciable. This seems unavoidable for the development of the theory and in particular for the determination of the normal modes of eq. (1) that in turn is a necessary prerequisite for the standard quantization of the field. That problem has been solved only for spin 1 field in flat RW metric and for spin 1/2 field in all cases of the RW curvature parameter [13, 24]. As far as the author knows, the problem is still open for the other values of the spin.

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