Adv. Studies Theor. Phys., Vol. 4, 2010, no. 5, 233 - 239

Effective Potential of the $c = \frac{6}{7}$ Theory

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Abstract

Following our previous work on fractional spin symmetries (FSS), we develop in this work an hermitean spin1/3 supefield formulation of the $c = 6/7$ critical theory leading, once the auxiliary fields are eliminated, to an explicit form of a scalar potential $V(\varphi, \varphi^*)$. We show then that the flow to the $c = 4/5$ Potts model is achieved by a spontaneous breaking of the off critical (1/3.1/3) symmetry.

To contribute much more to the subject of fractional spin symmetries (FSS) [1, 2, 3, 4, 5], which deals with exotic particles, we shall fix our attention in this work on the following subalgebra

$$\begin{aligned} Q^{-3} &= P, \\ \overline{Q^{+3}} &= \overline{P}, \\ Q^{-}\overline{Q^{+}} - q\overline{Q^{+}}Q^{-} &= \Delta \end{aligned} \tag{1}$$

generated by $\left(Q^{-}, \overline{Q}^{+}, P, \overline{P} \text{ and } \Delta\right)$. As noted earlier, the remarkable feature of this algebra is its invariance under the $Z_3 \times Z_2$ automorphism group acting as:

$$\begin{aligned} \overline{Q}^{+} &= (Q^{-})^{*},\ \overline{P} = P^{*}, \\ \Gamma\overline{Q}^{+} &= q\overline{Q}^{+}, \Gamma Q^{-} = q^{2}Q^{-}, \Gamma P = P. \end{aligned} \tag{2}$$

Introducing the real spin fractional superspace $\left(z, \theta^{+}, \overline{z}, \overline{\theta^{+}}, x\right)$ with $\theta^{+3} = \overline{\theta}^{-3} = 0$, the superalgebra Eq(1) is realized as .

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$$
\begin{aligned}
Q^{-} &= \frac{\partial}{\partial\theta^{+}} + \theta^{+2}\frac{\partial}{\partial z} + \alpha\overline{\theta}^{-}\frac{\partial}{\partial x}, \\
\overline{Q}^{+} &= \frac{\partial}{\partial\overline{\theta}^{-}} + \overline{\theta}^{-2}\frac{\partial}{\partial\overline{z}} + \alpha\theta^{+}\frac{\partial}{\partial x}, \\
P &= -\overline{q}\frac{\partial}{\partial z}, \quad \overline{P} = -q\frac{\partial}{\partial z} \\
Q &= \alpha\left(1-q\right)\frac{\partial}{\partial x}
\end{aligned}
\tag{3}
$$

These relations may be checked by following the same lines as did previously in [6, 7, 8]. Note that the undesired terms vanish by using the nilpotency condition $\theta^{-3} = \overline{\theta}^{+3} = 0$, the identity $1 + q + \overline{q} = 0$ and by the commutation rule $\overline{\theta}^{-}\theta^{+} = \overline{q}\theta^{+}\overline{\theta}^{-}$. Superfields ϕ defined on the real spin 1/3 superspace, we are considering in this work, are functions depending on the coordinates $z, \overline{z}, \theta^{+}, \overline{\theta}^{-}$ and eventually on x. They are off shell representations of Eqs (1) which may carry both a spin and a Z_3 charge $n = 0, \pm 1$. A generic superfield of weight $(h, \overline{h})$ and Z_3 charge n is developable in a finite series as

$$
\begin{aligned}
\phi^{-} &= \varphi^{-} + \theta^{+}\psi^{+} + \overline{\theta}^{-}\eta^{0} + \theta^{+}\overline{\theta}F^{-} + \theta^{+2}\chi^{0} \\
&\quad + \overline{\theta}^{-2}\mu^{+} + \theta^{+2}\overline{\theta}^{-}\nu^{+} + \overline{\theta}^{-2}\theta^{+}\xi^{0} + \theta^{+2}\overline{\theta}^{-2}D^{-}.
\end{aligned}
\tag{4}
$$

The conformal weights of the component fields of ϕ^{n} are easily deduced by knowing that the weights of θ^{+} and $\overline{\theta}^{-}$ are $(-1/3, 0)$ and $(0, -1/3)$ respectively. For instance, the scale dimension of F^{n} and D^{n} are $\left(h + 1/3, \overline{h} + 1/3\right)$ and $\left(h + 2/3, \overline{h} + 2/3\right)$ respectively. Their Z_3 charge is n. Note also that superfields type Eq(4) can be subject to a reality condition. Note also that the highest θ-component of ϕ^{n} namely D^{n} transforms as a total space time derivative under the change $\delta\theta^{+} = \varepsilon^{+}$and $\delta\overline{\theta}^{-} = \overline{\varepsilon}^{-}$. Invariant actions S are then constructed as in (1/2.1/2) supersymmetric theories

$$
S = \int d^{2}z d^{2}\overline{\theta}^{-} d^{2}\overline{\theta}^{+} L,
\tag{5}
$$

where the super-Lagrangian L is Z_3 neutral and scales as a $1/3 + 1/3$ dimensional quantity. In this above equation we have ignored the x-dependence realizing the topological charge. This can be ensured by supposing that the superfields involved in L are x-independent and that x parametrizes one dimensional compact manifold. Taking as dynamical superfields ϕ^{-} and its conjugate $\phi^{+} = (\phi^{-})^{\dagger}$ and using dimensional arguments and convariance, it is not difficult to see that the superfield lagrangian form reads as:

$$L \sim D^{-}\phi\overline{D}^{+}\phi^{*} + W(\phi,\phi^{*}), \tag{6}$$

where $W(\phi,\phi^{*})$ is a hermitean superpotential whose dependence in the superfields ϕ and ϕ^{*} is dictated by the two dimensional $(1/3,1/3)$ supersymmetric model we want to describe. Taking ϕ as in Eqs(4) and its conjugate ϕ^{*} as follows

$$\begin{aligned} \phi^{+} &= \varphi^{+} + \theta^{+}\overline{\eta}^{0} + \overline{\theta}^{-}\overline{\psi}^{-} + q\theta^{+}\overline{\theta}^{-}F^{*} + \theta^{+2}\overline{\mu}^{-} \\ &+\overline{\theta}^{-2}\overline{\chi}^{0} + \theta^{+2}\overline{\theta}^{-}\overline{\xi}^{0} + \overline{\theta}^{-2}\theta^{+}\overline{\nu}^{-} + q\theta^{+2}\overline{\theta}^{-2}D^{*}, \end{aligned} \tag{7}$$

the derivatives $D^{-}\phi$ and $\overline{D}^{+}\phi^{*}$ are then

$$\begin{aligned} D^{-}\phi &= (\psi^{+} + \overline{\theta}^{-}F + q^{2}\overline{\theta}^{-2}\xi^{0}) - \overline{q}\theta^{+}(\chi^{0} + \overline{\theta}^{-}\nu^{+} + \overline{\theta}^{-2}D) \\ &+\theta^{+2}(\partial\varphi + \overline{\theta}^{-}\partial\eta^{0} + \overline{\theta}^{-2}\partial\mu^{+}), \end{aligned} \tag{8}$$

and

$$\begin{aligned} \overline{D}^{+}\phi^{*} &= (\overline{\psi}^{-} + \theta^{+}F^{+} + q\theta^{+2}\overline{\xi}^{0}) - q\overline{\theta}^{-}(\overline{\chi}^{0} + \theta^{+}\overline{\nu}^{-} + \theta^{+2}D^{*}) \\ &+\overline{\theta}^{-2}(\overline{\partial}\varphi^{*} + \theta^{+}\overline{\partial}\overline{\eta}^{0} + \theta^{+2}\overline{\partial}\overline{\mu}^{-}), \end{aligned} \tag{9}$$

Using these relations and integrating with respect to $d^{2}\theta$, the $(1/3,1/3)$ supersymmetric invariant component field kinetic terms read as

$$\begin{aligned} L &\sim \partial\varphi\overline{\partial}\varphi^{*} + q\psi^{+}\overline{\partial}\overline{\mu}^{-} + \overline{q}\partial\mu^{+}\overline{\psi}^{-} - \partial\eta^{0}\overline{\chi}^{0} - \overline{\partial}\overline{\eta}^{0}\chi^{0} \\ &-qDF^{*} - \overline{q}FD^{*} + q\overline{\xi}^{0}\xi^{0} - q\nu^{+}\overline{\nu}^{-}. \end{aligned} \tag{10}$$

Note that this equation contains two kinds of fields. Dynamical fields $\varphi, \psi^{+}, \eta^{0}, \chi^{0}$ and μ^{+} together with their complex conjugates. The solutions of their equations of motion is

$$\begin{aligned} \varphi &= \varphi(z) + \varphi(\overline{z}) \\ \psi^{+} &= \psi^{+}(z), \overline{\psi}^{-} = \overline{\psi}^{-}(\overline{z}) \\ \mu^{+} &= \mu^{+}(\overline{z}), \overline{\mu}^{-} = \overline{\mu}^{-}(z) \\ \chi^{0} &= \chi^{0}(z), \overline{\chi}^{0} = \overline{\chi}^{0}(\overline{z}) \\ \eta^{0} &= \eta^{0}(\overline{z}), \overline{\eta}^{0} = \overline{\eta}^{0}(z) \end{aligned} \tag{11}$$

So that the free field propagators are:

$$\begin{aligned} \langle \varphi(z_1)\, \varphi^*(z_2) \rangle &\sim \ln z_{12} \\ \left\langle \psi^+(z_1) \overline{\mu}^-(z_2) \right\rangle &\sim z_{12}^{-1} \\ \left\langle \chi^0(z_1) \overline{\eta}^0(z_2) \right\rangle &\sim z_{12}^{-1} \end{aligned} \tag{12}$$

The second kind of fields involved in L_0 are the auxiliary fields. These are the scalar fields F and D and their complex conjugates F^* and D^* which will play a special role in the derivation of the scalar potential $V(\varphi, \varphi^*)$. The other auxiliary fields ξ^0 and ν^+ and their conjugates $\overline{\xi}^0$ and $\overline{\nu}^-$ are parafermionic fields of spin $s = -1/3, 1/3, 1/3$ and $-1/3$ respectively. They are not important in our present study and then we shall forget about them. We turn now to examine the superpotential W that we take for the moment as

$$W(\phi, \phi^+) = \sum \lambda_n \phi^n + cc. \tag{13}$$

Setting all non scalar fields to zero in the superfields ϕ and ϕ^*, for a matter of simplicity, and integrating with respect to $d^4\theta$ we get

$$U(\varphi, F, D) = \sum \lambda_n \left[nD\phi^{n-1} + q\frac{n(n-1)}{2} F^2 \phi^{n-1} \right] + cc, \tag{14}$$

or equivalently by using the field derivatives of the superpotentail W

$$U(\varphi, F, D) = \frac{\partial W}{\partial \varphi} D + \frac{q}{2} \frac{\partial^2 W}{\partial \varphi^2} F^2 + cc, \tag{15}$$

where we have set $\frac{\partial W}{\partial \phi} = \frac{\partial W}{\partial \varphi}$ with θ^+ and θ^- taken equal to zero. Combining this result with the expression of the kinetic terms L_0, the full component lagrangian $L = L_0 + L_{int}$ of Eqs(10) and (15) reads therefore as

$$L = L_0 + \left[\left(\frac{\partial W}{\partial \varphi} D + \frac{q}{2} \frac{\partial^2 W}{\partial \varphi^2} F^2 \right) + cc \right]. \tag{16}$$

Note that the auxiliary fields D and D^* appear usually linearly in the above Lagrangian. These are Lagrange fields leading to a couple of constraint equations relating the remaining scalar fields of the theory. Moreover they allow to solve the auxiliary fields F and F^* in terms of the dynamical fields φ and φ^*. Indeed, the equations of motions of D and D^*, which read easily from Eq(16), imply:

$$F = q\frac{\partial W}{\partial \varphi^*}, \overline{F} = \overline{q}\frac{\partial W}{\partial \varphi}, \tag{17}$$

so that the scalar potential $V(\varphi, \varphi^*)$ we are looking for is completely expressed in terms of F and F^* as:

$$V(\varphi, \varphi^*) = \frac{q^2}{2}\frac{\partial(F^* F^2)}{\partial\varphi} + cc, \tag{18}$$

or equivalently by using their field equations Eq(17)

$$V(\varphi, \varphi^*) = \frac{1}{2}\frac{\partial^2 W}{\partial\varphi^2}\left(\frac{\partial W}{\partial\varphi^*}\right) + cc. \tag{19}$$

Before going ahead recall that the Hamiltonian $H = \left(P + \overline{P}\right)/2$ of this theory reads in terms in terms of the spin 1/3 superymmetric generators Q^- and $\overline{Q}^+$, as:

$$2H = \left(Q^{-3} + \overline{Q}^{+3}\right),$$

which is not necessary a positive operator although it is hermitean. If $|\psi\rangle$ is a given state of the underlying Hilbert space, then the energy eigenvalue $2E = \langle\psi| Q^{-3} |\psi\rangle + \langle\psi| \overline{Q}^{+3} |\psi\rangle$ is not in general positive. Supposing that H is bounded below, ie that there exists a fundamental state $|\psi_0\rangle$ of eigenvalue E_0 such that $E \geq E_0$, then we have the two following statements: **1**-If $E_0 = 0$, then the $s = 1/3$ supersymmetry is preserved and the minimum of $V(\varphi, \varphi^*)$ is zero. **2**-If $E_0 > 0$, then the spin $s = 1/3$ supersymmetry is spontaneously broken if and only if:

$$\frac{\partial W}{\partial\varphi} \neq 0; \frac{\partial^2 W}{\partial\varphi^2} \neq 0. \tag{20}$$

Using the expression of the superpotential W, we find that the above constraints are solved by:

$$\langle D\rangle \neq 0; \left\langle F^2\right\rangle \neq 0, \tag{21}$$

or equivalently $\lambda_1, \lambda_2 \neq 0$. These equations are the necessary and sufficient conditions for spontaneous spin 1/3 supersymmetry breaking. Note that both the fields D and F^2 behave exactly as the conformal field of the $c = 6/7$ critical theory. Thus, as in the $c = 7/10(1/2, 1/2)$ supersymmetric theory, here also the $\phi_{1,3}$ perturbation breaks the fractional symmetry of the TPM. Such feature is expected to hold for all $D = 2(1/k, 1/k)$ supersymmetric theories. We illustrate herbelow this result on the example $W = g(\phi + \phi^{*n})/n$ which by help of Eq (19) leads to the scalar potential:

$$V(\varphi,\varphi^*)=\frac{(n-1)}{2}g^3\,(\varphi\varphi^*)^{n-2}\,(\varphi^n+\varphi^{*n}). \tag{22}$$

For n a multiple of three, $n=3p$; this potential becomes $Z_3\times Z_3$ invariant. Setting $\varphi=\rho\exp(i\sigma)$, where ρ and σ are two real fields with $\rho(z,\overline{z})\geq 0$ and $0\leq\sigma(z,\overline{z})\leq\pi$, Eq (23) may be be rewritten as:

$$V(\rho,\sigma)=(n-1)g^3\rho^{3n-4}\cos(n\sigma). \tag{23}$$

The positivity condition of this potential depends on the values of $\sigma(z,\overline{z})$ and the sign of the coupling constant g. $V(\rho,\sigma)$ is positive if $g>0$ and $0\leq n\sigma\leq\pi/2$ or again $g<0$ and $\pi/2\leq n\sigma\leq\pi$. Consider now the perturbed superpotential $g\phi^n/n+\delta\phi^n/2+\lambda\phi+cc$. The new expression of the scalar potential reads as:

$$V(\varphi,\varphi^*)=\frac{1}{2}(\delta+ng\varphi^{n-2})(\lambda+\delta\varphi^*+g\varphi^{*(n-1)})+cc. \tag{24}$$

Under the above hypothesis, the minimum of this perturbed potential is no longer zero since $V_{\min}=\lambda.\delta$ thus the $D=2(1/3,1/3)$ supersymmetry is broken by the D and F^2 terms as predicted by Eqs(21-22).

In the remainder of this work, we extend the above results to an arbitrary superpotential of the from:

$$W(\phi,\phi^*)=\sum_{n,m}\lambda_{n,m}\phi^n\phi^{*m}+cc. \tag{25}$$

Using Eqs (4) and (7) by dropping out all non scalar fields and integrating with respect to $d^4\theta$, we get a new lagrangian generalizing Eqs(1):

$$\begin{aligned} L &= L_0+\frac{\partial W}{\partial\varphi}D+\frac{\partial W}{\partial\varphi^*}D^*+\frac{q}{2}\frac{\partial^2 W}{\partial\varphi^2}F^2 \\ &\quad +\frac{\overline{q}}{2}\frac{\partial^2 W}{\partial\varphi^{*2}}F^{*2}+\frac{\partial^2 W}{\partial\varphi\partial\varphi^*}FF^*, \end{aligned} \tag{26}$$

where L_0 is given by Eq (6). Choosing W to be the sum of an analytic and antianalytic functions of the superfields ϕ and ϕ^* as in Eq (13), we recover the previous result. Eliminating the lagrangian fields D and D^* through their equations of motions (17), we get the scalar potential $V(\varphi,\varphi^*)$:

$$V(\varphi,\varphi^*)=\frac{1}{2}\frac{\partial^2 W}{\partial\varphi^2}\left(\frac{\partial W}{\partial\varphi^*}\right)^2+\frac{1}{2}\frac{\partial^2 W}{\partial\varphi^{*2}}\left(\frac{\partial W}{\partial\varphi}\right)^2+\frac{\partial^2 W}{\partial\varphi\partial\varphi^*}\left(\frac{\partial W}{\partial\varphi}\right)\left(\frac{\partial W}{\partial\varphi^*}\right) \tag{27}$$

Under the same hypothesis as for the superpotential Eq(13), the spontaneous breaking of the $D = 2(1/3, 1/3)$ supersymmetry occurs for $\langle D \rangle \neq 0$ and $\langle F^2 \rangle \neq 0$ or again $\langle D \rangle \neq 0$ and $\langle FF^* \rangle \neq 0$. As an example we consider the superpotential $W = \gamma\varphi\varphi^* + g(\varphi^3 + \varphi^{*3})/3$. The resulting Z_3 invariant component scalar potential $V(\varphi, \varphi^*)$ follows from Eq (28):

$$V(\varphi, \varphi^*) = 2g\gamma^3(\varphi^3 + \varphi^{*3}) + g\gamma(g + 4\gamma)\varphi\varphi^* + \varphi\varphi^* \left[\gamma^3 + g^3(\varphi^3 + \varphi^{*3})\right]. \tag{28}$$

As expected this potential vanishes for $\varphi = 0$. Perturbing this model by the $D-$term $\lambda \int d^4\theta\phi + cc$, we find that the value of $V(\varphi, \varphi^*)$ at $\varphi = \varphi^* = 0$ is no longer zero as it is proportional to $\lambda.\gamma$. Two dimensional $(1/3, 1/3)$ supersymmetry is then spontaneously broken.

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Received: October, 2009