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Integrability of q-Deformed Lax Equations

A. EL Boukili and M. B. Sedra[1]

Université Ibn Tofail, Faculté des Sciences
Département de Physique, Laboratoire de Physique de la
Matière et Rayonnement (LPMR), Kénitra, Morocco

Abstract

The theory of q-deformed pseudo-differential operators can be defined by means of the q-derivative ∂_q instead of the usual derivative ∂ with respect to the one dimensional space coordinate x in a classical system. Based on previous works on pseudo-differential operators [arXiv:0708.4046 and hep-th/0610056], we focus in this work to present the basics steps towards constructing the q-deformed integrable systems and the associated Lax generating technique. Particular interest is devoted the the q-Burgers and the q-KdV systems and their underlying mapping.

1 q-pseudo-differential operators

The q-differential operator ∂_q acts on the ring of functions as follows [1],

$$\partial_q \circ f = (\partial_q f) + \eta_q(f)\partial_q, \tag{1}$$

where the q-shift operator η_q is a linear function of f given by

$$\eta_q(f(x)) = f(qx). \tag{2}$$

For non-local differential operators, we have

$$\partial_q^{-1} \circ f = \sum_{k\succeq 0}(-1)^k q^{-k(k+1)/2}(\eta_q^{-k-1}(\partial_q^k f)\partial_q^{-k-1}. \tag{3}$$

This equation is obtained by using the following relation

$$(\partial_q^{-1} \circ \partial_q) \circ f = (\partial_q \circ \partial_q^{-1}) \circ f = f. \tag{4}$$

[1]mysedra@yahoo.fr

where ∂_q^{-1} is the inverse of the formal q-derivative ∂_q. We should note that η_q does not commute with ∂_q by virtue of,

$$\partial_q(\eta_q^k(f)) = q^k \eta_q^k(\partial_q f), \qquad k \in \mathbb{Z}. \tag{5}$$

The general expression is given by

$$\partial_q^m(\eta_q^k(f)) = q^{km} \eta_q^k(\partial_q^m f), \qquad k, m \in \mathbb{Z}. \tag{6}$$

Note that eq.(1) can be generalized as follows

$$\partial_q^n \circ f = \sum_{k \geq 0} \binom{n}{k}_q \eta_q^{n-k}(\partial_q^k f)\partial_q^{n-k}, \tag{7}$$

for all n. In the last equation, the q-binomials take the form

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q....(n-k+1)_q}{(1)_q(2)_q...(k)_q}, \tag{8}$$

and the q-numbers are given by

$$(n)_q = \frac{q^n - 1}{q - 1}, \tag{9}$$

with $\binom{n}{0}_q = 1$.
We can write out several explicit forms of (7) for q-derivative ∂_q^n and ∂_q^{-n} as follows

$$\begin{aligned}
\partial_q \circ f &= (\partial_q f) + \eta_q(f)\partial_q, \\
\partial_q^2 \circ f &= (\partial_q^2 f) + (q+1)\eta_q(\partial_q f)\partial_q + \eta_q^2(f)\partial_q^2, \\
\partial_q^3 \circ f &= (\partial_q^3 f) + (q^2+q+1)\eta_q(\partial_q^2 f)\partial_q + (q^2+q+1)\eta_q^2(\partial_q f)\partial_q^2 + \eta_q^3(f)\partial_q^3,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\partial_q^{-1} \circ f &= \eta_q^{-1}(f)\partial_q^{-1} - q^{-1}\eta_q^{-2}(\partial_q f)\partial_q^{-2} + q^{-3}\eta_q^{-3}(\partial_q^2 f)\partial_q^{-3} - q^{-6}\eta_q^{-4}(\partial_q^3 f)\partial_q^{-4} \\
&\quad + \frac{1}{q^{10}}\eta_q^{-5}(\partial_q^4 f)\partial_q^{-5} + \cdots + (-1)^k q^{-(1+2+3+\cdots+k)}\eta_q^{-k-1}(\partial_q^k f)\partial_q^{-k-1} + \cdots
\end{aligned} \tag{11}$$

$$\begin{aligned}
\partial_q^{-2} \circ f &= \eta_q^{-2}(f)\partial_q^{-2} - \frac{1}{q^2}(2)_q\eta_q^{-3}(\partial_q f)\partial_q^{-3} + \frac{1}{q^{(2+3)}}(3)_q\eta_q^{-4}(\partial_q^2 f)\partial_q^{-4} \\
&\quad - \frac{1}{q^{(2+3+4)}}(4)_q\eta_q^{-5}(\partial_q^3 f)\partial_q^{-5} + \cdots \\
&\quad + \frac{(-1)^k}{q^{(2+3+\cdots+k+1)}}(k+1)_q\eta_q^{-2-k}(\partial_q^k f)\partial_q^{-2-k} + \cdots
\end{aligned}$$

2 q-deformed Lax generating technique

We focus here to present some results related to the Lax representation in its q-deformed version by using the convention notations and the analysis presented in previous occasions [5, 6]. The obtained results are shown to be compatible with the ones already established in literature.

2.1 q-differential Lax equation

The basic idea of the Lax formulation consists first in considering a q-integrable system with the Lax representation

$$[\mathcal{L}, \partial_t - B]_q = 0, \tag{12}$$

where
▷ The time derivative is $\partial_t \equiv \frac{\partial}{\partial t}$,
▷ The deformed Lie Bracket $[.,.]_q$ is defined as follows [11]

$$\begin{aligned} [f,g]_q &= f \circ g - q^{|f|.|g|} \ f \circ g \qquad (13)\\ &= f \circ g - q^{4[f].[g]} \ f \circ g. \qquad (14) \end{aligned}$$

where $|f|$ is the grading of the function f .

▷ The Equation (12) is called q-differential Lax equation and the associated pair of operators $(\mathcal{L}, B)$ are called the Lax q-differential operators.

▷ The q-differential operator $\mathcal{L}$ defines the integrable system which we should fix from the beginning.

The $sl_n - KdV$ hierarchy in the q-deformed version is defined as

$$\frac{\partial \mathcal{L}}{\partial t_k} = [(\mathcal{L}^{\frac{k}{2}})_+, \mathcal{L}]_q. \tag{15}$$

where L is a differential operator of grading (conformal weight) 2. Equivalently on may write

$$[\mathcal{L}, \partial_t - B]_q \equiv [\mathcal{L}, \partial_{t_k} - (\mathcal{L}^{\frac{k}{2}})_+]_q = 0 \tag{16}$$

where the operator B is the analogue of $(\mathcal{L}^{\frac{k}{2}})_+$ describing a q-differential operator of grading k.

Now, let us apply the q-deformed Lax-pair generating technique. We need to find an appropriate operator B satisfying the equation (12), for this we have to make some constraints on the operator B namely

Ansatz for the operator B:

$$B = \partial_q^n \circ \mathcal{L}^m + \tilde{B}, \tag{17}$$

with ∂_q^n is the q-differential operator acting on $\mathcal{L}^m$ according to equation (17) and $\tilde{B}$ is another operator such that $|B|=|\tilde{B}|$. Then, with this ansatz, the problem reduces to find the operator $\tilde{B}$.

To understand the situation, we will study two interesting examples namely the $q - KdV$ and the $q - Burgers$ systems in the particular situation where $q^8 = 1$, ie:

$$q \in \left\{1, \frac{\sqrt{2}}{2}(1+i), i, \frac{\sqrt{2}}{2}(1+i), -1, \frac{\sqrt{2}}{2}(1+i), -i, \frac{\sqrt{2}}{2}(1+i)\right\}. \tag{18}$$

2.2 q-deformed Burgers equations

The $\mathcal{L}$-operator for the q-deformed Burgers equation is given by

$$\mathcal{L}_{q-burgers} = \partial_q + u_1 \tag{19}$$

with

$$\mathcal{L}_{q-burgers} \in \widetilde{\mathcal{A}}_1^{(0,1)} \tag{20}$$

For $n = 1 = m$, the anzats for the q-deformed Burgers operator B give

$$\begin{aligned} B &= \partial_q \circ \mathcal{L} + \tilde{B} \\ &= \partial_q^2 + \eta_q(u_1)\partial_q + \partial_q(u_1) + \tilde{B} \end{aligned} \tag{21}$$

Simply algebraic computations lead to

$$\begin{aligned} \left[\mathcal{L}, \tilde{B}\right]_q &= (\eta_q(u_1) - u_1)\partial_q^2 \\ &+ \left[q\eta_q(\partial_q(u_1)) + (\eta_q(u_1))^2 + \partial_q(u_1) - \partial_q(\eta_q(u_1)) - u_1\eta_q(u_1)\right]\partial_q \\ &+ \eta_q(u_1)\partial_q(u_1) - \dot{u}_1 \end{aligned} \tag{22}$$

where $\dot{u}_1 = \frac{\partial u_1}{\partial t}$.

Now, our goal is to extract, from the equations (12) and (22), the Lax equation which is called q-deformed Burgers. For this we will follow the following procedure:

Ansatz for the operator $\tilde{B}$:

$$\tilde{B} = \alpha\partial_q + \beta \tag{23}$$

where α and β are arbitrary functions on u and its derivatives. one finds

$$\begin{aligned}\left[\mathcal{L},\tilde{B}\right] = & \left(\eta_q\left(\alpha\right)-q^8\alpha\right)\partial_q^2 \\ & +\left[\partial_q\left(\alpha\right)+\eta_q\left(\beta\right)+u_1\alpha-q^8\alpha\eta_q\left(u_1\right)-q^8\beta\right]\partial_q \\ & +\partial_q\left(\beta\right)-q^8\alpha\partial_q\left(u_1\right)+u_1\beta-q^8\beta u_1\end{aligned} \quad (24)$$

giving rise to

$$a\partial_q^2\left(u_1\right)+\left(b-1\right)\left[\eta_q\left(u_1\right)\partial_q\left(u_1\right)+u_1\partial_q\left(u_1\right)\right]+\dot{u}_1=0 \quad (25)$$

for arbitrary constants a and b.

The Classical Limit:

The equation (25) is called *q-deformed Burgers equation* or simply the *q-Burgers equation.* Note the fact that this equation is linear for $b=1$ and for $q=1$ (ie. $\eta_q\left(u_1\right)=u_1$) we recover the same equation gotten in works [10] and [6] namely

$$au_1''+2\left(b-1\right)u_1u_1'+\dot{u}_1=0. \quad (26)$$

2.3 q-deformed KdV equations

We consider here the q-deformed KdV Lax operator

$$\mathcal{L}_{q-KdV}=\partial_q^2+u_2. \quad (27)$$

We are going to follow the same method of the previous example. Therefore the Ansatz for the operator B is

$$\begin{aligned}B &= \partial_q\circ\mathcal{L}+\tilde{B} \\ &= \partial_q^3+\eta_q\left(u_2\right)\partial_q+\partial_q\left(u_2\right)+\tilde{B}\end{aligned} \quad (28)$$

and the associated Lax equation

$$\left[\partial_t-B,\mathcal{L}\right]_q=0 \quad (29)$$

or equivalently

$$\left[\mathcal{L},B\right]_q=-\dot{u}_1 \quad (30)$$

Straighforward computations give the following $q-KdV$ equation

$$\dot{u}_2=\left[u_2+\eta_q\left(u_2\right)\right]\partial_q\left(u_2\right)+\partial_q^2\left[\partial_q\left(u_2\right)+\eta_q\left(\partial_qu_2\right)\right] \quad (31)$$

The Classical Limit:

Setting $q=1$, we recover the standard KdV equation[7] namely

$$\dot{u}_2=u_2u_2'+u_2''' \quad (32)$$

3 q-Burgers$\hookrightarrow$q-KdV mapping

We present here an approach to define the correspondence between the q-Burgers and the q-KdVsystems. Such a correspondence that we call the *q-Burger-KdV mapping* is shown to generalize the standard Burgers-KdV mapping studied in a previous work [6].

We illustrate this idea for the KdV and Burgers equations and present later a generalization for the sl_n case.

Now consider the Burgers q-differential operator (19)

$$\mathcal{L}_{q-burgers} = \partial_q + u_1 \in \widetilde{\mathcal{A}}_1^{(0,1)} \tag{33}$$

and the KdV q-differential operator (27)

$$\mathcal{L}_{q-KdV} = \partial_q^2 + u_2 \in \widetilde{\mathcal{A}}_2^{(0,2)} / \widetilde{\mathcal{A}}_2^{(1,1)} \tag{34}$$

Proposition 1:

We have the following decomposition rule

$$\mathcal{L}_{q-KdV}\left(u_2\right) = \mathcal{L}_{q-burgers}\left(u_1\right) \circ \mathcal{L}_{q-burgers}(v_1) \tag{35}$$

where we have set $v_1 = -\eta_q\left(u_1\right)$ and $u_2 = \partial_q\left(-\eta_q\left(u_1\right)\right) - u_1\eta_q\left(u_1\right)$.

This decomposition is called q-deformed Miura transformation. Equivalently we have

$$\mathcal{L}_{q-burgers}\left(u_1\right) \hookrightarrow \mathcal{L}_{q-KdV}\left(u_2\right) = \mathcal{L}_{q-burgers}\left(u_1\right) \circ \mathcal{L}_{q-burgers}\left(-\eta_q\left(u_1\right)\right) \tag{36}$$

with $u_2 = \partial_q\left(-\eta_q\left(u_1\right)\right) - u_1\eta_q\left(u_1\right)$

Proposition 2:

On the basis of the following grading rules $[\partial_{t_{q-KdV}}] = 3$ and $\left[\partial_{t_{q-Burgers}}\right] = 2$, we can set

$$\partial_{t_{q-Burgers}} \hookrightarrow \partial_{t_{q-KdV}} = \alpha\partial_q \circ \partial_{t_{q-Burgers}} + \beta\partial_q^3 \tag{37}$$

where α and β are arbitrary real constants.

Proposition 3: Generalization

Given a q-deformed Burgers operator $\mathcal{L}_{q-Burgers}$ and a q-deformed $\mathrm{sl}_n - KdV$ operator of type

$$\mathcal{L}_{q-sl_n-KdV} = \partial_q^n + u_2\partial_q^{n-2} + u_3\partial_q^{n-3} + ... + u_n \tag{38}$$

then we can make the following decomposition

$$\mathcal{L}_{q-sl_n-KdV} = \mathcal{L}_{q-Burgers}(v_1) \circ \mathcal{L}_{q-Burgers}(v_2) \circ .. \circ \mathcal{L}_{q-Burgers}(v_n) \tag{39}$$

where v_i, $i = 1, .., n$ are fields of conformal weight i and which are functions of the fields u_j, $j = 2, .., n$ and their q-derivatives.

References

[1] C. Kassel, Commun. Math. Phys. 140 (1992)343.

[2] C.H. Oh and K. Singh, Conformal Properties of Primary Fields in a q-Deformed Theory, arXiv:hep-th/9407081v2 21 Jul 1994.

[3] J.Wess, q-deformed Heisenberg Algebras, arXiv:math-ph/9910013v1 8 Oct 1999.

[4] Jingsong HE, Yinghua LI and Yi Cheng, q-Deformed KP Hierarchy and q-Deformed Constrained KP Hierarchy, arXiv:nlin/0606039v2 [nlin.SI] 15 Jun 2006.

[5] A. El Boukili, M.B. Sedra and A. Zemate, Integrable Systems: Bosonic and Supersymmetric Cases, African Journal Of Mathematical Physics Volume 6 (2008) 123-143.

[6] M.B. Sedra, Moyal noncommutative integrability and the Burgers-KDV mapping, Nucl.Phys.B740:243-270,2006, hep-th/0508236; A. El Boukili, E.H. Saidi and M.B. Sedra, Zeeman Effect In The Framework of Moyal Noncommutativity and String Theory, hep-th/0610123.

[7] O. Dafounansou, A. El Boukili and M.B. Sedra, Some aspects of noncommutative integrable systems, hep-th/0508173.

[8] Omer F. Dayi,q–Deformed Star Products Andmoyal Brackets, q-alg/9609023

[9] Richard Herrmann, q-deformed Lie algebras and fractional calculus, arXiv:0711.3701v1 [physics.gen-ph] 23 Nov 2007

[10] A. Jellal and E.H. El Kinani, Graded q-pseudodifferential operators and supersymmetric algebras, Journal of Physics A, Mathematical and General 35 No 16, 2002, pp 3697-3702.

[11] Viktor Abramov, On a graded q-differential algebra, JNMP Vol. 13, Supp. (2006), 1–8

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