

Validity of Born Approximation for Nuclear Scattering in Path Integral Representation

M. R. Pahlavani and R. Morad

Department of Physics, Faculty of Science, Mazandaran University
P.O. Box 47415-416, Babolsar, Iran
m.pahlavani@umz.ac.ir, r.morad@umz.ac.ir

Abstract

The first and second Born approximation are studied with the path integral representation for \mathcal{T} matrix. The \mathcal{T} matrix is calculated for Woods-Saxon potential scattering. To make corresponding integrals solvable analytically, an approximate function for the Woods-Saxon potential is used. Finally it is shown that the Born series converges at high energies and orders higher than two in Born approximation series can be neglected.

Keywords: Path Integral, Scattering, \mathcal{S} Matrix, Born Approximation, Woods-Saxon Potential

1 Introduction

The path integrals are very useful in some complicated problems such as quantum revolution, quantum field theory, etc, because of its compact operational notation with all the extent of finite-dimensional integral calculus like integration by parts, analytic continuation stationary phases approximation, etc[1,2,3,4,5]. For the example, in quantum field theory, both relativistic and non-relativistic, the path integrals have an important role. Since, they provide a relatively easy road to quantization and to expressions for Green's functions[6,7,8], which are closely related to amplitudes for physical processes such as scattering and decays of particles. Furthermore, the close relation between statistical mechanics and quantum mechanics, or statistical field theory and quantum field theory, is plainly visible via path integrals. Now, path-integral formalism is widely used in many branches of theoretical physics and in particular in nuclear physics[9]. The existence of a strong nuclear force in atomic nuclei revealed by the exceptional role of the nuclear magic number provides the foundation of the nuclear shell model. This strong force is believed to

be approximated most closely by a Woods-Saxon potential [10] either from analyzing the radial dependence of the nuclear central force or by deriving it from a microscopic two body force acted in neutron proton scattering [11]. The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, was successful to deduce the nuclear energy levels [12]. Also it was used as central part for the interaction of neutron with heavy nucleus [13]. The Woods-Saxon potential was used as a part of optical model in elastic scattering of some ions with heavy target in low range of energies [14]. In this paper we will use the path integral representation of the \mathcal{T} matrix in potential scattering for driving the different orders of Born approximation with the Woods-Saxon potential. This representation is not a phase-space path integral but it is a particular path integral over velocities that reduced the complexity of path-integral[15]. In the limit of large scattering times where energy conservation, the "dangerous" phases from \mathcal{S} matrix are created. The "Phantom" degrees of freedom is utilized to eliminate these phases. In addition, energy conservation is applied by imposing a Faddeev-Popov-like constraint in the velocity path integral[15]. In Sec.II we outline the \mathcal{T} matrix representation from the velocity path integrals. In order to obtain different terms of Born approximation, we use path integral representation for the scattering in presence of Woods-Saxon potential in Sec.III. In this investigation we show that the Born approximation is a converge series, therefore in elastic scattering analysis, we can neglect the orders higher than two in Born series.

2 Path Integrals for the \mathcal{T} matrix

In the framework of non-relativistic potential scattering, consider a central potential, $V(r)$ that vanishes at infinity. \mathbf{k}_i and \mathbf{k}_f are the initial and final momentum of a particle with mass m (see Figure.1). In this calculation, supposed the scattering states are normalized and $\hbar = 1$ [15], so

$$\langle \varphi_f | \varphi_i \rangle = (2\pi)^3 \delta^3(k_i - k_f). \quad (1)$$

The \mathcal{S} matrix is the matrix element of the evolution operator in the interaction picture that taken between scattering states and calculated at asymptotic times:

$$\mathcal{S}_{i \rightarrow f} = \lim_{T \rightarrow \infty} \langle \phi_f | \hat{U}_I(T, -T) | \phi_i \rangle = \lim_{T \rightarrow \infty} e^{i(E_i + E_f)T} \langle \phi_f | \hat{U}(T, -T) | \phi_i \rangle, \quad (2)$$

where the $\hat{U}_I(T, -T)$ is the time evolution operator that is defined as follows

$$\hat{U}_I(t_b, t_a) = e^{i\hat{H}_0 t_b} \exp \left[-i\hat{H}(t_b - t_a) \right] e^{-i\hat{H}_0 t_a} . \quad (3)$$

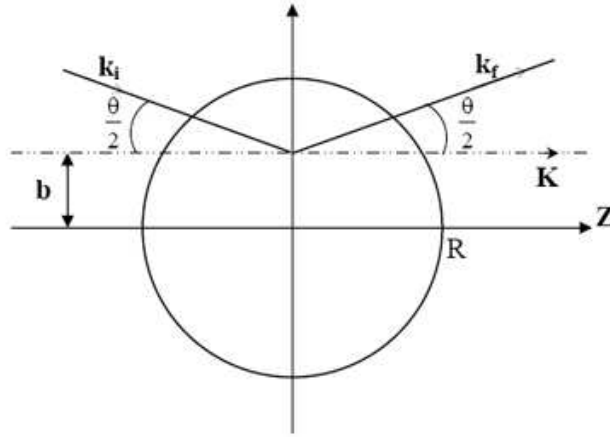


Figure 1: Scattering geometry for a potential of radius R , the impact parameter b . Incoming and outgoing momenta are $\mathbf{k}_{i,f}$, and the mean momentum is $\mathbf{K} = (\mathbf{k}_i + \mathbf{k}_f)/2$.

The \mathcal{T} matrix is defined usually by subtract the identity from \mathcal{S} matrix and factor out an energy conserving Dirac delta function

$$\mathcal{S}_{i \rightarrow f} = (2\pi)^3 \delta^{(3)}(\mathbf{k}_i - \mathbf{k}_f) - 2\pi i \delta(E_i - E_f) \mathcal{T}_{i \rightarrow f}, \quad (4)$$

where $E_i = E_f = E = k^2/(2m)$ is the common scattering energy. By integrating functionally over velocities instead of paths[16] the following formula is achieved[15]

$$\begin{aligned} (\mathcal{S} - 1)_{i \rightarrow f} &= \lim_{T \rightarrow \infty} \exp\left(\frac{i\mathbf{q}^2 T}{4m}\right) \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} \mathcal{N}^3 \int \mathcal{D}^3v \exp\left[i \int_{-T}^{+T} dt \frac{m}{2} \mathbf{v}^2(t)\right] \\ &\times \left\{ \exp\left[-i \int_{-T}^T dt V\left(\mathbf{r} + \frac{\mathbf{K}}{m}t + \mathbf{x}_v(t)\right)\right] - 1 \right\} \end{aligned} \quad (5)$$

where $\mathcal{N}^3 = \mathcal{N}^3(T, -T)$ is defined

$$\mathcal{N}(t_a, t_b) := \left(\int \mathcal{D}v \exp\left[i \int_{t_a}^{t_b} dt \frac{m}{2} v^2(t)\right] \right)^{-1}, \quad (6)$$

and

$$\mathbf{x}_v(t) = \frac{1}{2} \int_{-T}^{+T} dt' \operatorname{sgn}(t - t') \mathbf{v}(t'). \quad (7)$$

The momentum transfer and the mean momentum are defined by

$$\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i, \quad \mathbf{K} = \frac{1}{2} (\mathbf{k}_i + \mathbf{k}_f). \quad (8)$$

It can be shown that the 'dangerous phases' $q^2T/4m$ that are proportional to T , are canceled in each order of perturbation theory by introducing 'Phantom' degrees of freedom(dynamical variables with the wrong sign kinetic term)[15], so that the limit $T \rightarrow \infty$ can indeed be applied. Then the following path-integral representation for the \mathcal{S} matrix is obtained[15],

$$\begin{aligned}
 (\mathcal{S} - 1)_{i \rightarrow f} &= \lim_{T \rightarrow \infty} \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} |\mathcal{N}(T, -T)|^6 \\
 &\times \int \mathcal{D}^3v \mathcal{D}^3w \exp \left[i \int_{-T}^{+T} dt \frac{m}{2} (\mathbf{v}^2(t) - \mathbf{w}^2(t)) \right] \\
 &\times \left\{ \exp \left[-i \int_{-T}^{+T} dt V \left(\mathbf{r} + \frac{\mathbf{K}}{m}t + \mathbf{x}_v(t) - \mathbf{x}_w(0) \right) \right] - 1 \right\} \quad (9)
 \end{aligned}$$

where $\mathbf{w}(t)$ is a three-dimensional 'antivelocitv'. This is similar to the Lee-Wick approach to quantum Electrodynamics where they introduced the fields with a wrong sign kinetic term to remove all infinities[17,18]. To extract the \mathcal{T} matrix from the \mathcal{S} matrix for weak interaction, we can develop in powers of the potential and factor out in each order an energy conserving δ function. But to achieve this without a perturbative expansion of \mathcal{S} matrix usually the trick which Faddeev and Popov have introduced in field theory for quantization of non-Abelian gauge theories is used [19] and the following expression for the \mathcal{T} matrix is obtained[15],

$$\begin{aligned}
 \mathcal{T}_{i \rightarrow f}^{(3-3)} &= i \frac{K}{m} \int d^2b e^{-i\mathbf{q}\cdot\mathbf{b}} |\mathcal{N}|^6 \int \mathcal{D}^3v \mathcal{D}^3w \exp \left\{ i \int_{-\infty}^{+\infty} dt \frac{m}{2} [\mathbf{v}^2(t) - \mathbf{w}^2(t)] \right\} \quad (10) \\
 &\times \left\{ e^{i\chi_{\mathbf{K}}(\mathbf{b}, \mathbf{v}, \mathbf{w})} - 1 \right\}.
 \end{aligned}$$

Here the limit $T \rightarrow \infty$ is taken and the corresponding Gaussian normalization factor is written as

$$\mathcal{N} := \mathcal{N}(+\infty, -\infty). \quad (11)$$

In this equation the phase $\chi_{\mathbf{K}}$ is defined as

$$\chi_{\mathbf{K}}(\mathbf{b}, \mathbf{v}, \mathbf{w}) = - \int_{-\infty}^{+\infty} dt V \left(\mathbf{b} + \frac{\mathbf{K}}{m}t + \mathbf{x}_v(t) - \mathbf{x}_w(0) - \lambda \hat{\mathbf{K}} \right), \quad (12)$$

where \mathbf{b} is the impact parameter. θ is the scattering angle, one may obtain

$$q \equiv |\mathbf{q}| = 2k \sin \left(\frac{\theta}{2} \right), \quad K \equiv |\mathbf{K}| = k \cos \left(\frac{\theta}{2} \right). \quad (13)$$

For $t_0 = 0$ that is the most symmetric choice the gauge parameter, $\lambda = Kt_0/m$, is zero. The superscript "3-3" indicates that there are three-dimensional anti-velocity that used to cancel divergent phases in the limit of asymptotic times moreover the three- dimensional velocity variables. Then by expanding the exponent in powers of the potential the complete Born series is reproduced from this path-Integral representation as follows[15],

$$\mathcal{T}_{i \rightarrow f} =: \sum_{n=1}^{\infty} \mathcal{T}_n, \quad (14)$$

with

$$\begin{aligned} \mathcal{T}_n^{(3-3)} = & i \frac{K}{m} \frac{(-i)^n}{n!} \int d^2b e^{-i\mathbf{q}\cdot\mathbf{b}} \prod_{i=1}^n \left(\int_{-T}^{+T} dt_i \int \frac{d^3p_i}{(2\pi)^3} \tilde{V}(\mathbf{p}_i) \right) \\ & \times \exp \left\{ i \sum_{i=1}^n \mathbf{p}_i \cdot [\mathbf{b} + \mathbf{x}_{\text{ref}}(t_i)] \right\} G_n^{(3-3)}. \end{aligned} \quad (15)$$

Where $G_n^{(3-3)}$ is evaluated as

$$G_n^{(3-3)} = \exp \left\{ -\frac{i}{4m} \sum_{i,j=1}^n \mathbf{p}_i \cdot \mathbf{p}_j (T - |t_i - t_j| - T) \right\}, \quad (16)$$

and

$$\mathbf{x}_{\text{ref}}(t) = \frac{\mathbf{K}}{m} t. \quad (17)$$

$\tilde{V}(\mathbf{p}_i)$ is the fourier transform of potential

$$\tilde{V}(\mathbf{p}_i) = \int d^3r V(r) e^{i\mathbf{q}\cdot\mathbf{r}}. \quad (18)$$

3 Born Approximation for Woods-Saxon Potential

In this section, we calculated the first and the second order of Born approximation from the path integral formula for \mathcal{T} matrix. For first Born approximation from the Eq.(15) we have

$$\mathcal{T}_1^{(3-3)} = \frac{K}{m} \int d^2b e^{-i\mathbf{q}\cdot\mathbf{b}} \int_{-T}^{+T} dt \int \frac{d^3p}{(2\pi)^3} \tilde{V}(\mathbf{p}) \times \exp \left\{ i\mathbf{p} \cdot \left(\mathbf{b} + \frac{\mathbf{K}}{m} t \right) \right\}. \quad (19)$$

That with integration over \mathbf{b} and t , we obtain

$$\mathcal{T}_1^{(3-3)} = \tilde{V}(\mathbf{p}_{\perp} = \mathbf{q}, \mathbf{p}_{\parallel} = 0) = \tilde{V}(\mathbf{q}). \quad (20)$$

Where \mathbf{p}_\perp and \mathbf{p}_\parallel are the components of \mathbf{p} perpendicular and parallel to \mathbf{K} respectively. Also \mathbf{q} is a vector in the plane which is perpendicular to \mathbf{K} . The second Born approximation is calculated in the following formula

$$\begin{aligned} \mathcal{T}_2^{(3-3)} = & \frac{-iK\pi}{m} \int \int d^3p_1 d^3p_2 \tilde{V}(\mathbf{p}_1) \tilde{V}(\mathbf{p}_2) \delta^{(2)}(\mathbf{q} - \mathbf{p}_{1\perp} - \mathbf{p}_{2\perp}) \\ & \times \delta\left(\mathbf{p}_1 \cdot \frac{\mathbf{K}}{m} + \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{2m}\right) \delta\left(\mathbf{p}_2 \cdot \frac{\mathbf{K}}{m} - \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{2m}\right). \end{aligned} \quad (21)$$

To simplify the above equation, we suppose that $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$, then the Eq.(21) can be rewritten as

$$\mathcal{T}_2^{(3-3)} = \frac{-im\pi}{K} \int d^2p \tilde{V}(\mathbf{p}) \tilde{V}(\mathbf{q} - \mathbf{p}). \quad (22)$$

Integration over \mathbf{p} can be done in the plane perpendicular to \mathbf{K} . Now, we apply this formalism to calculate the first and second Born approximation for Woods-Saxon potential. The Woods-Saxon potential can be defined by

$$V(r) = \frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)}. \quad (23)$$

This potential may be represented, with precision better than 3% for any r value (see figure.2) by

$$V(r) = \frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)} = V_0 C\left(\frac{r-R_0}{a}\right) \quad (24)$$

$$C(x \leq 0) = 1 - \frac{7}{8} e^x + \frac{3}{8} e^{2x} \quad (25)$$

$$C(x \geq 0) = e^{-x} \left(1 - \frac{7}{8} e^{-x} + \frac{3}{8} e^{-2x} \right). \quad (26)$$

This approximation is particularly useful in obtaining analytical expression for integrals that involve the Woods-Saxon potential. A such approximation is used for tow parameter Fermi distribution density function[20]. With this approximation the fourier transform of Woods-Saxon potential is

$$\tilde{V}(\mathbf{q}) = \pi V_0 a^3 \left[\frac{7e^{-R_0/a}}{(q^2 a^2 + 1)^2} - \frac{6e^{-2R_0/a}}{(q^2 a^2 + 4)^2} \right], \quad (27)$$

where q is the magnitude of \mathbf{q} . And the second Born approximation can be written as,

$$\mathcal{T}_2^{(3-3)} = \frac{m\pi^3 V_0^2 a^6}{K} [49 e^{-2R_0/a} g_1(q) - 42 e^{-3R_0/a} g_2(q) + 36 e^{-4R_0/a} g_3(q)], \quad (28)$$

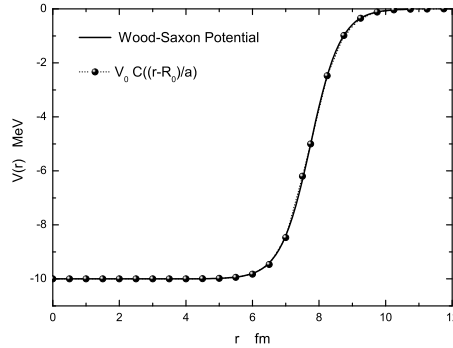


Figure 2: Nuclear potential for the $^{16}\text{O}+^{20}\text{Ne}$ scattering represented by Woods-Saxon potential(solid line) and the approximate function $V_0 C((r - R_0)/a)$ (symbol \bullet).

where $g_1(q)$, $g_2(q)$ and $g_3(q)$ are solvable integrals over p , but their results are too complicated, thus their results are not appeared here. It should be noted that q is a function of scattering energy (E) and angle(θ), therefore

$$g_1(q) = \int dp \frac{p}{(p^2 a^2 + 1)^2 ((q - p)^2 a^2 + 1)^2}, \quad (29)$$

$$g_2(q) = \int dp p \left[\frac{1}{(p^2 a^2 + 1)^2 ((q - p)^2 a^2 + 4)^2} + \frac{1}{(p^2 a^2 + 4)^2 ((q - p)^2 a^2 + 1)^2} \right], \quad (30)$$

$$g_3(q) = \int dp \frac{p}{(p^2 a^2 + 4)^2 ((q - p)^2 a^2 + 4)^2}. \quad (31)$$

We wish to calculate the first and second Born approximation for the $^{16}\text{O} + ^{20}\text{Ne}$. The parameters of Woods-Saxon potential for this system are presented in table.1. The first and second Born approximation, for this reaction, as a function of scattering energy and angle are shown in Figure.3-a and Figure.3-b respectively. The graphs show that the values of T_1 and T_2 are significant in small angles and extremely reduced with increasing of angle. For comparison between first and second Born approximation, we show these approximation versus the scattering angle θ , at $E=24.5$ MeV in Figure.4. It can be seen from this Figure that the value of T_2 is smaller than T_1 and can be neglected. Also in fig.5 the values of T_1 and T_2 for $\theta = 0^\circ, \theta = 45^\circ, \theta = 90^\circ$ versus scattering energy are plotted. These graphs show that the value of T_2 , in all angles and

in energies below and near the coulomb barrier, is very smaller than T_1 . This is arises from fact that the Woods-Saxon is a short ranged potential. In low energies the repulsive coulomb force prevents the nuclei to be close. So it is not convenient to use the nuclear potential to study the scattering. In this energies, the scattering is Rutherford scattering. Finally we can conclude that the Born approximation is a converge series and we can neglect higher order of Born approximation in elastic scattering studies.

Table 1: Woods-Saxon parameters for $^{16}O + ^{20}Ne$ reaction. ($R_0 = r_0(A_1^{1/3} + A_2^{2/3})$) [21].

<i>System</i>	$V_0(MeV)$	$r_0(fm)$	$a(fm)$
$^{16}O + ^{20}Ne$	10	1.48	0.45

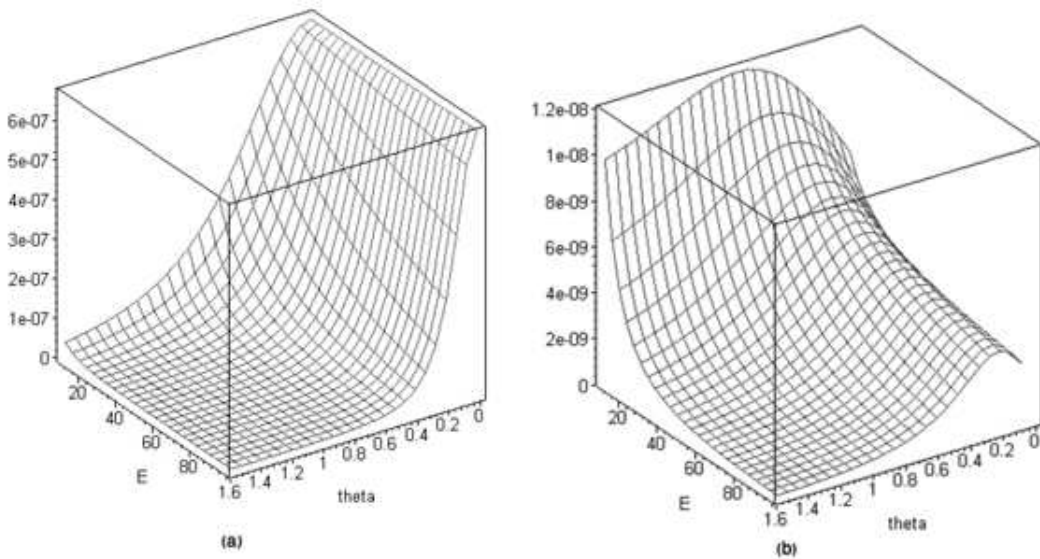


Figure 3: a) First Born approximation and b) Second Born approximation versus scattering energy and angle(θ) for $^{16}O + ^{20}Ne$ and Woods-Saxon potential.

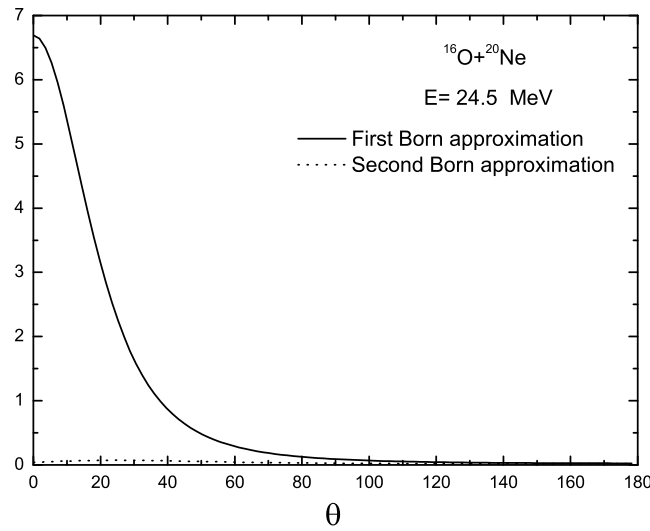


Figure 4: The first and second Born approximation calculated with path integral formalism for $^{16}\text{O} + ^{20}\text{Ne}$ scattering versus θ in $E = 24.5$ MeV. solid line shows the first Born approximation and dash line shows the second Born approximation. (in scale 10^{-7}).

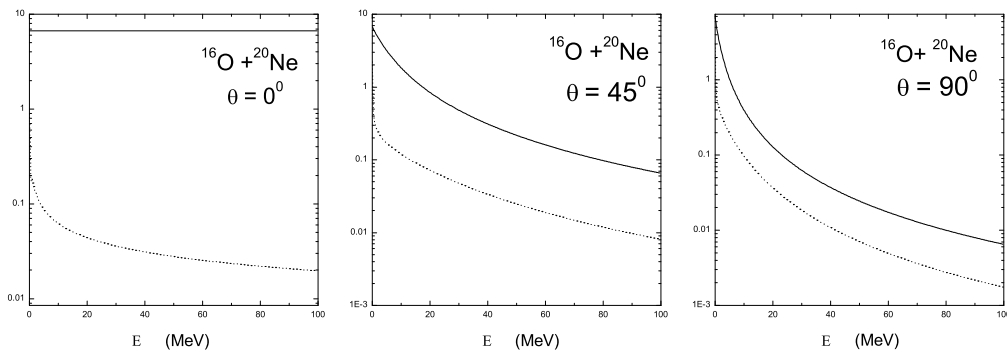


Figure 5: The first and second Born approximation calculated with path integral formalism for $^{16}\text{O} + ^{20}\text{Ne}$ scattering versus scattering energy (E) in three angle : $\theta = 0^\circ$, $\theta = 45^\circ$ and $\theta = 90^\circ$. solid line shows the first Born approximation and dash line shows the second Born approximation. (in scale 10^{-7}).

References

- [1] A. Dynin, A rigorous path integral construction in any dimension, *Letters in Mathematical Physics*, **62** (1998), 317.
- [2] W. Zheng-Dong, Mathai-Quillen Formalism - From the Point of View of Path Integrals, *Letters in Mathematical Physics*, **39** (1997), 331 - 340.
- [3] T.J. Zastawniak, Fresnel type path integrals for the stochastic Schroedinger equation, *Letters in Mathematical Physics*, **41** (1997), 93-99.
- [4] T. Ichinose, H. Tamura, Note on the Norm Convergence of the Unitary Trotter Product Formula, *Letters in Mathematical Physics*, **70** (2004) 65-81.
- [5] V.S. Varadarajan, Path Integrals for a Class of P-Adic Schrödinger Equations, *Letters in Mathematical Physics*, **39** (1997), 97-106.
- [6] D.J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, World Scientific, 1984.
- [7] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985.
- [8] P. Ramond, *Field Theory: A Modern Primer*, Addison-Wesley, 1990.
- [9] D.M. Brink, U. Smilanski, Multiple Reflections in the Path-Integral Approach to Barrier Penetration, *Nuclear physics A*, **405** (1983), 301-312.
- [10] R.D. Woods and D. S. Saxon, Diffuse surface optical model for nucleon-nuclei scattering, *Phys. Rev.* **95** (1954), 577-587.
- [11] A. Bohr and B.R. Mottelson, *Nuclear Structure Vol. 1*, Benjamin New York, 1969.
- [12] J.M.G. Gomez, K. Kar, V.K.B. Kota, R.A. Molina and J. Retamosa, Localisation in 2p1f nuclear shell model wavefunctions, *Phys. Lett. B*, **567** (2003), 251-258.
- [13] H. Nicolai, Supersymmetry and spin systems, *J. Phys. A: Math. Gen.*, **9** (1976), 1497-1506.
- [14] O. V. Bepulova, E. A. Romanovsky and T. I. Spasskaya, Nucleon-nucleus real potential of Woods-Saxon shape between -60 and +60 MeV for the 40 A 208 nuclei, *Journal of Phys. G : Nuclear and particle physics*, **29** (2003), 1193-1211.

- [15] R. Rosenfelder, Path integrals for potential scattering, *Phys. Rev. A*, **79** (2009), 012701.
- [16] W. da Cruz, Functional integral over velocities for a spinning particle with and without anomalous magnetic moment in a constant electromagnetic field, *J. Phys. A*, **30** (1997), 5225.
- [17] B. Grinstein and D. O'Connell, One-loop renormalization of Lee-Wick gauge theory, *Phys. Rev. D*, **78** (2008), 105005.
- [18] T. D. Lee and G. C. Wick, Negative Metric and the Unitarity of the S Matrix, *Nucl. Phys. B*, **9** (1969), 209.
- [19] M. Levy and J. Sucher, Eikonal Approximation in Quantum Field Theory, *Phys. Rev.* **186** (1969), 1656.
- [20] Chamon et al, Toward a global description of the nucleus-nucleus interaction, *Phys. Rev. C*. **66** (2002), 014610.
- [21] A. A. Farra, A Study of Heavy-Ion Elastic Scattering Reactions Using a Semimicroscopic Model, *Tr. J. of Physics*, **22** (1998), 895-900.

Received: October, 2009