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Reformulation of Max Born's Theory for Information Science

Koji Nagata[1] and Tadao Nakamura[2,3,4]

[1] Obihiro University of Agriculture and Veterinary Medicine
ko_mi_na1@yahoo.co.jp

[2] Imperial College, London Department of Computing
[3] Stanford University, Computer Systems Laboratory
[4] Keio University of Science and Technology

Abstract

It is shown that there is a contradiction within conventional quantum mechanics [K. Nagata, Int J Theor Phys (2009), DOI: 10.1007/s10773-009-0158-z] based on the Born statistical formula. We investigate whether the Born statistical formula meets our physical world. We derive a proposition concerning a quantum expectation value under the assumption of the existence of the orientation of reference frames in N spin-1/2 systems. This assumption intuitively depictures our physical world. The quantum predictions within the formalism of the Born statistical formula violate the proposition with a magnitude that grows exponentially with the number of particles. The Born statistical formula cannot depicture our physical world with a violation factor that grows exponentially with the number of particles. The Born statistical formula cannot meet our physical world. We reformulate the Born statistical formula.

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1 Introduction

It is shown that there is a contradiction within conventional quantum mechanics [1] based on the Born statistical formula [2, 3]. The Born statistical formula (cf. [4, 5, 6, 7, 8, 9]) gives approximate and at times remarkably accurate numerical predictions. Much experimental data approximately fits to the quantum predictions of the Born statistical formula for the past some 100 years.

The Born statistical formula says new science with respect to information theory. The science is called the quantum information theory [9]. The Born statistical formula gives us very useful another theory in order to create new information science and to explain the handing of raw experimental data in our physical world.

As for the foundations of the Born statistical formula, Leggett-type nonlocal variables theory [10] is experimentally investigated [11, 12, 13]. The experiments report that the Born statistical formula does not accept Leggett-type nonlocal variables interpretation. As for the applications of the Born statistical formula, there are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm [14] with polarization and transverse spatial modes of the electromagnetic field as qubits [15]. Single-photon Bell states are prepared and measured [16]. The decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [17]. A one-way based experimental implementation of Deutsch's algorithm is reported [18].

The Born statistical formula seems successful formula and it looks no problem in order to use it experimentally. Several researches address [4] the mathematical formulation of the quantum theory. The Born statistical formula is accepted widely. It is desirable that the Born statistical formula is mathematically successful because we predict unknown physical phenomena precisely. Sometimes such predictions are effective in the field of elementary particle physics. We endure much time in order to see the fact by using for example large-scale accelerator. Rolf Landauer says Information is Physical [9]. We cannot create any computer without physical phenomena. This fact motivates us to investigate the Hilbert space formalism of the Born statistical formula. Here we ask. Does the Born statistical formula depicture our physical world? Unfortunately it is not so even in both the macroscopic scale and the microscopic scale. The theoretical formalism of the implementation of the Deutsch-Jozsa algorithm [14, 19] relies on the Born statistical formula. We cannot implement the Deutsch-Jozsa algorithm by using the Born statistical formula.

In this paper we investigate whether the Born statistical formula meets our physical world. We derive a proposition concerning a quantum expectation value under the assumption of the existence of the orientation of reference frames in N spin-1/2 systems. This assumption intuitively depictures our physical world. The quantum predictions within the formalism of the Born statistical formula violate the proposition with a magnitude that grows exponentially with the number of particles. The Born statistical formula cannot depicture our physical world with a violation factor that grows exponentially with the number of particles. The Born statistical formula cannot meet our physical world. We reformulate the Born statistical formula.

In what follows we confine ourselves to the two-level (e.g. electron spin, photon polarizations, and so on). In what follows we confine ourselves to the discrete eigenvalue case.

2 the Born statistical formula

Let **R** denote the reals. We assume that every eigenvalue in this paper lies in **R**. We assume that every Hermitian operator is associated with a unique observable (see Ref. [6]). We do not need to distinguish between them in this paper.

We investigate whether the Born statistical formula meets our physical world. Let $\mathcal{O}$ be the space of Hermitian operators described in a Hilbert space. Let $\mathcal{T}$ be the space of density operators described in the Hilbert space. Namely $\mathcal{T} = \{\rho|\rho \in \mathcal{O} \wedge \rho \geq 0 \wedge \mathrm{Tr}[\rho] = 1\}$. We define the notation r which represents one result of quantum measurements. Suppose that measurement of a Hermitian operator A for a system in a state ρ yields a value $r(A) \in \mathbf{R}$. We consider the following proposition. We define $\chi_\Delta(x)$ as the characteristic function. We assume $x \in \mathbf{R}$. We define Δ as any subset of the reals **R**.

Proposition: *The Born statistical formula*

$$\mathrm{Prob}(\Delta)^{\rho}_{r(A)} = \mathrm{Tr}[\rho\chi_\Delta(A)]. \tag{1}$$

The symbol $(\Delta)^{\rho}_{r(A)}$ denotes the following proposition. $r(A)$ lies in Δ if the system is in a state ρ. The symbol "Prob" denotes a probability that the proposition $(\Delta)^{\rho}_{r(A)}$ holds.

The possible value of $r(A)$ takes eigenvalues of A if we assign the truth value "1" for the Born statistical formula.

3 Problem of the Born statistical formula

3.1 Plane settings model

3.1.1 Original Born statistical formula

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is a spin-1/2 pure state lying in the z-x plane. Assume that one source of N uncorrelated spin-carrying particles emits them in a state which can be described as a multi spin-1/2 pure uncorrelated state. Parameterize the settings of the jth observer with a unit vector $\vec{n}_j$ (its direction along which the spin component is measured) with $j = 1, \ldots, N$. We can introduce the 'Born' correlation function which is the average of the product of the results of measurements

$$E_{\mathrm{Born}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N)\rangle_{\mathrm{avg}} \tag{2}$$

where r is the result. The value of r is ± 1 (in $(\hbar/2)^N$ unit) which is obtained if the measurement directions are set at $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$ if we accept the Born statistical formula.

Also we can introduce a quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state

$$E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \mathrm{Tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}]. \tag{3}$$

$\otimes$ denotes the tensor product. $\cdot$ denotes the scalar product in $\mathbf{R}^2$. $\vec{\sigma} = (\sigma_z, \sigma_x)$ denotes a vector of two Pauli operators. ρ denotes a multi spin-1/2 pure uncorrelated state

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \tag{4}$$

with $\rho_j = |\Psi_j\rangle\langle\Psi_j|$. $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the z-x plane. We can write the observable (unit) vector $\vec{n}_j$ in a plane coordinate system as follows

$$\vec{n}_j(\theta_j^{k_j}) = \cos\theta_j^{k_j}\vec{x}_j^{(1)} + \sin\theta_j^{k_j}\vec{x}_j^{(2)} \tag{5}$$

where $\vec{x}_j^{(1)} = \vec{z}_j$ and $\vec{x}_j^{(2)} = \vec{x}_j$ are the Cartesian axes. The angle $\theta_j^{k_j}$ takes n values

$$\theta_j^1 = 0,\ \theta_j^2 = \frac{\pi}{n}, \ldots, \theta_j^{n-1} = \frac{(n-2)\pi}{n},\ \text{and}\ \theta_j^n = \frac{(n-1)\pi}{n}. \tag{6}$$

We derive a necessary condition to be satisfied by the quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state given in (3). In more detail we derive $(E_{\mathrm{QM}}, E_{\mathrm{QM}})$ which is the value of the scalar product of the quantum correlation function E_{QM} given in (3). We use decomposition (5). We introduce simplified notations as

$$T_{i_1 i_2 \ldots i_N} = \mathrm{Tr}[\rho\vec{x}_1^{(i_1)} \cdot \vec{\sigma} \otimes \vec{x}_2^{(i_2)} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{x}_N^{(i_N)} \cdot \vec{\sigma}] \tag{7}$$

and

$$\vec{c}_j = (c_j^1, c_j^2) = (\cos\theta_j^{k_j}, \sin\theta_j^{k_j}). \tag{8}$$

Then we have

$$(E_{\mathrm{QM}}, E_{\mathrm{QM}}) = \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left(\sum_{i_1,\ldots,i_N=1}^{2} T_{i_1 \ldots i_N} c_1^{i_1} \cdots c_N^{i_N} \right)^2 = \left(\frac{n}{2}\right)^N \sum_{i_1,\ldots,i_N=1}^{2} T_{i_1 \ldots i_N}^2 \leq \left(\frac{n}{2}\right)^N \tag{9}$$

where we use the orthogonality relation $\sum_{k_j=1}^{n} c_j^{\alpha} c_j^{\beta} = \frac{n}{2}\delta_{\alpha,\beta}$. The value of $\sum_{i_1,\ldots,i_N=1}^{2} T_{i_1 \ldots i_N}^2$ is bounded as $\sum_{i_1,\ldots,i_N=1}^{2} T_{i_1 \ldots i_N}^2 \leq 1$. We have

$$\prod_{j=1}^{N} \sum_{i_j=1}^{2} (\mathrm{Tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \leq 1. \tag{10}$$

It is important that the inequality (9) is saturated iff ρ is a multi spin-1/2 pure uncorrelated state such that $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the z-x plane for every j. The reason of the inequality (9) is due to the following quantum inequality

$$\sum_{i_j=1}^{2} (\mathrm{Tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \leq 1. \tag{11}$$

The inequality (11) is saturated iff $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the z-x plane. The inequality (9) is saturated iff the inequality (11) is saturated for every j. Thus we have the maximal possible value of the scalar product as a quantum proposition concerning our physical world

$$(E_{\rm QM}, E_{\rm QM})_{\max} = \left(\frac{n}{2}\right)^N. \tag{12}$$

when the system is in a multi spin-1/2 pure uncorrelated state.

On the other hand a correlation function satisfies the Born statistical formula if it can be written as

$$E_{\rm Born}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m} \tag{13}$$

where l denotes a label and r denotes the result of measurements of the dichotomic observables parameterized by directions of $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Assume the quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state given in (3) admits the Born statistical formula. We have the following proposition concerning the Born statistical formula

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m}. \tag{14}$$

In what follows we show that we cannot assign the truth value "1" for the proposition (14) concerning the Born statistical formula.

Assume the proposition (14) is true.

We have same quantum correlation function by changing the label l into l' and by changing the label m into m' as follows

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')}{m'}. \tag{15}$$

The value of the right-hand-side of (14) is equal to the value of the right-hand-side of (15) because we only change the labels.

We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)$ to $r(l)$. We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')$ to $r(l')$.

We have

$$
\begin{aligned}
&(E_{\rm QM}, E_{\rm QM}) \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(l)}{m} \times \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r(l')}{m'} \right) \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} r(l) r(l') \right) \\
&\leq \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} |r(l) r(l')| \right) \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_N=1}^{n} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} \right) = n^N.
\end{aligned} \tag{16}
$$

We use the following

$$|r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) \times r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')| = +1. \tag{17}$$

The inequality (16) is saturated since we have

$$
\begin{aligned}
&\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = 1 \wedge l \in \mathbf{N}\} = \{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = 1 \wedge l' \in \mathbf{N}\} \text{ and} \\
&\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = -1 \wedge l \in \mathbf{N}\} = \{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = -1 \wedge l' \in \mathbf{N}\}
\end{aligned} \tag{18}
$$

$\mathbf{N}$ denotes $\{1, 2, \ldots, +\infty\}$. Hence we have the following proposition concerning the Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = n^N. \tag{19}$$

We cannot assign the truth value "1" for two propositions (12) (concerning our physical world) and (19) (concerning the Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. Each of them is a spin-1/2 pure state lying in the z-x plane. We are in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state. We cannot accept the validity of the proposition (14) (concerning the Born statistical formula) if we assign the truth value "1" for the proposition (12) (concerning our physical world). In other words the Born statistical formula does not reveal our physical world.

3.1.2 Reformulation of the Born statistical formula

We reformulate the Born statistical formula. We have the maximal possible value of the scalar product as a quantum proposition concerning our physical world

$$(E_{\rm QM}, E_{\rm QM})_{\max} = \left(\frac{n}{2}\right)^N \tag{20}$$

when the system is in a multi spin-1/2 pure uncorrelated state. On the other hand we have the following proposition concerning the Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = n^N. \tag{21}$$

We cannot assign the truth value "1" for two propositions (20) (concerning our physical world) and (21) (concerning the Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. Each of them is a spin-1/2 pure state lying in the z-x plane. We are in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state.

We reformulate the Born statistical formula as follows.

Reformulated Born statistical formula:

$$\mathrm{Prob}(\Delta)^{\rho}_{r(A)} = \mathrm{Tr}[\rho\chi_{\Delta}(\frac{A}{\sqrt{2^N}})]. \tag{22}$$

The symbol $(\Delta)^{\rho}_{r(A)}$ denotes the following proposition. $r(A)$ lies in Δ if the system is in a state ρ. The symbol "Prob" denotes a probability that the proposition $(\Delta)^{\rho}_{r(A)}$ holds.

The proposition (21) (concerning the Born statistical formula) becomes the following new proposition concerning reformulated Born statistical formula when we accept reformulated Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = \left(\frac{n}{2}\right)^N. \tag{23}$$

We can assign the truth value "1" for both two propositions (20) (concerning our physical world) and (23) (concerning reformulated Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. Each of them is a spin-1/2 pure state lying in the z-x plane. We are not in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state. We solve the contradiction presented in the previous section by reformulating the Born statistical formula.

3.1.3 Relation between eigenvalues and quantum correlation function

E_A denotes a set of eigenvalues of A.

We have the following

$$\begin{aligned}
&E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) \\
&= \mathrm{Tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}] \\
&= \sqrt{2^N} \sum_{x \in \mathbf{R}} x \mathrm{Tr}[\rho \chi_{\{x\}}(\frac{\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}}{\sqrt{2^N}})] \\
&= \sum_{x \in \mathbf{R}} \sqrt{2^N} x \mathrm{Prob}(\{x\})^{\rho}_{r(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})} \\
&= \sum_{x \in \mathbf{R}} x \mathrm{Prob}(\{\frac{x}{\sqrt{2^N}}\})^{\rho}_{r(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})} \\
&= \sum_{x \in E_{\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}}} x \mathrm{Tr}[\rho \chi_{\{x\}}(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})]. \qquad (24)
\end{aligned}$$

We see our reformulation does not change conventional notation.

3.1.4 New type of the Deutsch-Jozsa algorithm

The earliest quantum algorithm, the Deutsch-Jozsa algorithm, is representative to show that quantum computation is faster than classical counterpart with a magnitude that grows exponentially with the number of qubits.

Let us follow the argumentation presented in [9]. —— The application, known as *Deutsch's problem*, may be described as the following game. Alice, in Amsterdam, selects a number x from 0 to $2^N - 1$, and mails it in a letter to Bob, in Boston. Bob calculates the value of some function $f : \{0, \ldots, 2^N - 1\} \to \{0, 1\}$ and replies with the result, which is either 0 or 1. Now, Bob has promised to use a function f which is of one of two kinds; either the value of $f(x)$ is constant for all values of x, or else the value of $f(x)$ is balanced, that is, equal to 1 for exactly half of all the possible x, and 0 for the other half. Alice's goal is to determine with certainty whether Bob has chosen a constant or a balanced function, corresponding with him as little as possible. How fast can she succeed?

In the classical case, Alice may only send Bob one value of x in each letter. At worst, Alice will need to query Bob at least $2^N/2 + 1$ times, since she may receive $2^N/2$ 0s before finally getting a 1, telling her that Bob's function is balanced. The best deterministic classical algorithm she can use therefore requires $2^N/2 + 1$ queries. Note that in each letter, Alice sends Bob N bits of information. Furthermore, in this example, physical distance is being used to artificially elevate the cost of calculating $f(x)$, but this is not needed in the general problem, where $f(x)$ may be inherently difficult to calculate.

If Bob and Alice were able to exchange qubits, instead of just classical bits, and if Bob agreed to calculate $f(x)$ using a unitary transformation U_f, then Alice could achieve her goal in just one correspondence with Bob, using the following algorithm.

Alice has an N qubit register to store her query in, and a single qubit register which she will give to Bob, to store the answer in. She begins by preparing both her query and answer registers in a superposition state. Bob will evaluate $f(x)$ using quantum parallelism and leave the result in the answer register. Alice then interferes states in the superposition using a Hadamard transformation (a unitary transformation), $H = (\sigma_z + \sigma_x)/\sqrt{2}$, on the query register, and finishes by performing a suitable measurement to determine whether f was constant or balanced.

Let us follow the quantum states through this algorithm. The input state is

$$|\psi_0\rangle = |0\rangle^{\otimes N}|1\rangle. \tag{25}$$

Here the query register describes the state of N qubits all prepared in the $|0\rangle$ state. After the Hadamard transformation on the query register and the Hadamard gate on the answer register we have

$$|\psi_1\rangle = \sum_{x\in\{0,1\}^N} \frac{|x\rangle}{\sqrt{2^N}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]. \tag{26}$$

The query register is now a superposition of all values, and the answer register is in an evenly weighted superposition of $|0\rangle$ and $|1\rangle$. Next, the function f is evaluated (by Bob) using $U_f : |x, y\rangle \to |x, y \oplus f(x)\rangle$, giving

$$|\psi_2\rangle = \pm \sum_x \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^N}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]. \tag{27}$$

Here $y \oplus f(x)$ is the bitwise XOR (exclusive OR) of y and $f(x)$. Alice now has a set of qubits in which the result of Bob's function evaluation is stored in the amplitude of the qubit superposition state. She now interferes terms in the superposition using a Hadamard transformation on the query register. To determine the result of the Hadamard transformation it helps to first calculate the effect of the Hadamard transformation on a state $|x\rangle$. By checking the cases $x = 0$ and $x = 1$ separately we see that for a single qubit $H|x\rangle = \sum_z (-1)^{xz}|z\rangle/\sqrt{2}$. Thus

$$\begin{aligned} &H^{\otimes N}|x_1, \ldots, x_N\rangle \\ &= \frac{\sum_{z_1,\ldots,z_N} (-1)^{x_1z_1+\cdots+x_Nz_N}|z_1, \ldots, z_N\rangle}{\sqrt{2^N}}. \end{aligned} \tag{28}$$

This can be summarized more succinctly in the very useful equation

$$H^{\otimes N}|x\rangle = \frac{\sum_z (-1)^{x\cdot z}|z\rangle}{\sqrt{2^N}}, \tag{29}$$

where $x \cdot z$ is the bitwise inner product of x and z, modulo 2. Using this equation and (27) we can now evaluate $|\psi_3\rangle$,

$$|\psi_3\rangle = \pm \sum_z \sum_x \frac{(-1)^{x\cdot z+f(x)}|z\rangle}{\sqrt{2^N}} \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]. \tag{30}$$

Alice now observes the query register. Note that the absolute value of the amplitude for the state $|0\rangle^{\otimes N}$ is $\sum_x (-1)^{f(x)}/2^N$. Let's look at the two possible cases — f constant and f balanced — to discern what happens. In the case where f is constant the absolute value of the amplitude for $|0\rangle^{\otimes N}$ is $+1$. Because $|\psi_3\rangle$ is of unit length it follows that all the other amplitudes must be zero, and an observation will yield $(+\frac{1}{\sqrt{2}})$s for all N qubits in the query register. Thus, global measurement outcome is $(+\frac{1}{\sqrt{2^N}})$. If f is balanced then the positive and negative contributions to the absolute value of the amplitude for $|0\rangle^{\otimes N}$ cancel, leaving an amplitude of zero, and a measurement must yield a result other than $+\frac{1}{\sqrt{2}}$ (i.e., $-\frac{1}{\sqrt{2}}$) on at least one qubit in the query register. Summarizing, if Alice measures all $(+\frac{1}{\sqrt{2}})$s and global measurement outcome is $(+\frac{1}{\sqrt{2^N}})$ the function is constant; otherwise the function is balanced.

We notice that the difference between $+\frac{1}{\sqrt{2^N}}$ and $-\frac{1}{\sqrt{2^N}}$ is approximately zero when $N \gg 1$. We question if the Deutsch-Jozsa algorithm in the macroscopic scale is possible or not. This question is open problem.

3.2 Sphere settings model

3.2.1 Original Born statistical formula

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is a spin-1/2 pure state. Assume that one source of N uncorrelated spin-carrying particles emits them in a state which can be described as a multi spin-1/2 pure uncorrelated state. Parameterize the settings of the jth observer with a unit vector $\vec{n}_j$ (its direction along which the spin component is measured) with $j = 1, \ldots, N$. We can introduce the 'Born' correlation function which is the average of the product of the results of measurements

$$E_{\text{Born}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N)\rangle_{\text{avg}} \tag{31}$$

where r is the result of measurements. The value of r is ± 1 (in $(\hbar/2)^N$ unit) which is obtained if the measurement directions are set at $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$ if we accept the Born statistical formula.

Also we can introduce a quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state

$$E_{\text{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \text{Tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}]. \tag{32}$$

$\otimes$ denotes the tensor product. $\cdot$ denotes the scalar product in $\mathbf{R}^3$. $\vec{\sigma} = (\sigma_z, \sigma_x, \sigma_y)$ denotes the vector of Pauli operator. ρ denotes a multi spin-1/2 pure uncorrelated state

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \tag{33}$$

with $\rho_j = |\Psi_j\rangle\langle\Psi_j|$. $|\Psi_j\rangle$ is a spin-1/2 pure state. We can write the observable (unit) vector $\vec{n}_j$ in a spherical coordinate system as follows

$$\vec{n}_j(\theta_j, \phi_j) = \sin\theta_j \cos\phi_j \vec{x}_j^{(1)} + \sin\theta_j \sin\phi_j \vec{x}_j^{(2)} + \cos\theta_j \vec{x}_j^{(3)} \tag{34}$$

where $\vec{x}_j^{(1)} = \vec{z}_j$, $\vec{x}_j^{(2)} = \vec{x}_j$, and $\vec{x}_j^{(3)} = \vec{y}_j$ are the Cartesian axes.

We derive a necessary condition to be satisfied by the quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state given in (32). In more detail we derive $(E_{\rm QM}, E_{\rm QM})$ which is the value of the scalar product of the quantum correlation function $E_{\rm QM}$ given in (32). We use decomposition (34). We introduce the measure $d\Omega_j = \sin\theta_j d\theta_j d\phi_j$ for the system of the jth observer. We introduce simplified notations as

$$T_{i_1 i_2 \ldots i_N} = {\rm Tr}[\rho \vec{x}_1^{(i_1)} \cdot \vec{\sigma} \otimes \vec{x}_2^{(i_2)} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{x}_N^{(i_N)} \cdot \vec{\sigma}] \tag{35}$$

and

$$\vec{c}_j = (c_j^1, c_j^2, c_j^3) = (\sin\theta_j \cos\phi_j, \sin\theta_j \sin\phi_j, \cos\theta_j). \tag{36}$$

Then we have

$$(E_{\rm QM}, E_{\rm QM}) = \int d\Omega_1 \cdots \int d\Omega_N \left(\sum_{i_1,\ldots,i_N=1}^{3} T_{i_1\ldots i_N} c_1^{i_1} \cdots c_N^{i_N} \right)^2 = \left(\frac{4\pi}{3}\right)^N \sum_{i_1,\ldots,i_N=1}^{3} T^2_{i_1\ldots i_N} \leq \left(\frac{4\pi}{3}\right)^N \tag{37}$$

where we use the orthogonality relation $\int d\Omega_j \, c_j^\alpha c_j^\beta = \frac{4\pi}{3}\delta_{\alpha,\beta}$. The value of $\sum_{i_1,\ldots,i_N=1}^{3} T^2_{i_1\ldots i_N}$ is bounded as $\sum_{i_1,\ldots,i_N=1}^{3} T^2_{i_1\ldots i_N} \leq 1$. We have

$$\prod_{j=1}^{N} \sum_{i_j=1}^{3} ({\rm Tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \leq 1. \tag{38}$$

It is important that this inequality is saturated iff ρ is a multi spin-1/2 pure uncorrelated state. The reason of the inequality (37) is due to the Bloch sphere

$$\sum_{i_j=1}^{3} ({\rm Tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \leq 1. \tag{39}$$

The inequality (39) is saturated iff $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state. The inequality (37) is saturated iff the inequality (39) is saturated for every j. Thus we have the maximal possible value of the scalar product as a quantum proposition concerning our physical world

$$(E_{\rm QM}, E_{\rm QM})_{\rm max} = \left(\frac{4\pi}{3}\right)^N \tag{40}$$

when the system is in a multi spin-1/2 pure uncorrelated state.

On the other hand a correlation function satisfies the Born statistical formula if it can be written as

$$E_{\rm Born}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m} \tag{41}$$

where l denotes a label and r denotes the result of measurements of the dichotomic observables parameterized by directions of $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Assume the quantum correlation function with the system in a multi spin-1/2 pure uncorrelated state given in (32) admits the Born statistical formula. We have the following proposition concerning the Born statistical formula

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m}. \tag{42}$$

In what follows we show that we cannot assign the truth value "1" for the proposition (42) concerning the Born statistical formula.

Assume the proposition (42) is true.

We have same quantum correlation function by changing the label l into l' and by changing the label m into m' as follows

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')}{m'}. \tag{43}$$

The value of the right-hand-side of (42) is equal to the value of the right-hand-side of (43) because we only change the labels.

We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)$ to $r(l)$. We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')$ to $r(l')$.

We have

$$\begin{aligned}
&(E_{\rm QM}, E_{\rm QM}) \\
&= \int d\Omega_1 \cdots \int d\Omega_N \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m} r(l)}{m} \times \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r(l')}{m'} \right) \\
&= \int d\Omega_1 \cdots \int d\Omega_N \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} r(l) r(l') \right) \\
&\leq \int d\Omega_1 \cdots \int d\Omega_N \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} |r(l) r(l')| \right) \\
&= \int d\Omega_1 \cdots \int d\Omega_N \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} \right) = (4\pi)^N.
\end{aligned} \tag{44}$$

We use the following

$$|r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) \times r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')| = +1. \tag{45}$$

The inequality (44) is saturated since we have

$$\begin{aligned}
&\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = 1 \wedge l \in \mathbf{N}\} = \{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = 1 \wedge l' \in \mathbf{N}\} \text{ and} \\
&\{l | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l) = -1 \wedge l \in \mathbf{N}\} = \{l' | r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l') = -1 \wedge l' \in \mathbf{N}\}
\end{aligned} \tag{46}$$

Hence we have the following proposition concerning the Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = (4\pi)^N. \tag{47}$$

We cannot assign the truth value "1" for two propositions (40) (concerning our physical world) and (47) (concerning the Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. Each of them is a spin-1/2 pure state. We are in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state. Thus we cannot accept the validity of the proposition (42) (concerning the Born statistical formula) if we assign the truth value "1" for the proposition (40) (concerning our physical world). In other words the Born statistical formula does not reveal our physical world.

3.2.2 Reformulation of the Born statistical formula

We reformulate the Born statistical formula. We have the maximal possible value of the scalar product as a quantum proposition concerning our physical world

$$(E_{\rm QM}, E_{\rm QM})_{\max} = \left(\frac{4\pi}{3}\right)^N \tag{48}$$

when the system is in a multi spin-1/2 pure uncorrelated state. On the other hand we have the following proposition concerning the Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = (4\pi)^N. \tag{49}$$

We cannot assign the truth value "1" for two propositions (48) (concerning our physical world) and (49) (concerning the Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. We are in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state.

We reformulate the Born statistical formula as follows.

Reformulated Born statistical formula:

$$\text{Prob}(\Delta)^{\rho}_{r(A)} = \text{Tr}[\rho \chi_{\Delta}(\frac{A}{\sqrt{3^N}})]. \tag{50}$$

The symbol $(\Delta)^{\rho}_{r(A)}$ denotes the following proposition. $r(A)$ lies in Δ if the system is in a state ρ. The symbol "Prob" denotes a probability that the proposition $(\Delta)^{\rho}_{r(A)}$ holds.

The proposition (49) (concerning the Born statistical formula) becomes the following new proposition concerning reformulated Born statistical formula when we accept reformulated Born statistical formula

$$(E_{\rm QM}, E_{\rm QM})_{\max} = \left(\frac{4\pi}{3}\right)^N. \tag{51}$$

We can assign the truth value "1" for both two propositions (48) (concerning our physical world) and (51) (concerning reformulated Born statistical formula) simultaneously when the system is in a multi spin-1/2 pure uncorrelated state. We are not in the contradiction when the system is in a multi spin-1/2 pure uncorrelated state. We solve the contradiction presented in the previous section by reformulating the Born statistical formula.

3.2.3 Relation between eigenvalues and quantum correlation function

E_A denotes a set of eigenvalues of A.

We have the following

$$\begin{aligned}
&E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) \\
&= \mathrm{Tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}] \\
&= \sqrt{3^N} \sum_{x \in \mathbf{R}} x \mathrm{Tr}[\rho \chi_{\{x\}}(\frac{\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}}{\sqrt{3^N}})] \\
&= \sum_{x \in \mathbf{R}} \sqrt{3^N} x \mathrm{Prob}(\{x\})^{\rho}_{r(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})} \\
&= \sum_{x \in \mathbf{R}} x \mathrm{Prob}(\{\frac{x}{\sqrt{3^N}}\})^{\rho}_{r(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})} \\
&= \sum_{x \in E_{\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}}} x \mathrm{Tr}[\rho \chi_{\{x\}}(\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma})].
\end{aligned} \tag{52}$$

We see our reformulation does not change conventional notation.

4 Conclusions

In conclusion we have investigated whether the Born statistical formula meets our physical world. We have derived a proposition concerning a quantum expectation value under the assumption of the existence of the orientation of reference frames in N spin-1/2 systems. The quantum predictions within the formalism of the Born statistical formula have violated the proposition with a magnitude that grows exponentially with the number of particles. The Born statistical formula cannot have depictured our physical world with a violation factor that grows exponentially with the number of particles. The Born statistical formula cannot have met our physical world. We have reformulated the Born statistical formula.

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References

[1] K. Nagata, Int J Theor Phys (2009), DOI: 10.1007/s10773-009-0158-z.

[2] M. Born, Zur Quantenmechanik der Stosvorgange, Zeitschrift fur Physik **37**, 863 (1926).

[3] M. Born, Quantenmechanik der Stosvorgange, Zeitschrift fur Physik **38**, 803 (1926).

[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[5] R. P. Feynman, R. B. Leighton, and M. Sands, *Lectures on Physics, Volume III, Quantum mechanics* (Addison-Wesley Publishing Company, 1965).

[6] M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Press, Oxford, 1989), 2nd ed.

[7] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).

[8] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley Publishing Company, 1995), Revised ed.

[9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).

[10] A. J. Leggett, Found. Phys. **33**, 1469 (2003).

[11] S. Gröblacher, T. Paterek, R. Kaltenbaek, Č. Brukner, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Nature (London) **446**, 871 (2007).

[12] T. Paterek, A. Fedrizzi, S. Gröblacher, T. Jennewein, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Phys. Rev. Lett. **99**, 210406 (2007).

[13] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, Phys. Rev. Lett. **99**, 210407 (2007).

[14] D. Deutsch, *Proc. Roy. Soc. London Ser. A* **400**, 97 (1985).

[15] A. N. de Oliveira, S. P. Walborn, and C. H. Monken, J. Opt. B: Quantum Semiclass. Opt. **7**, 288-292 (2005).

[16] Y.-H. Kim, Phys. Rev. A **67**, 040301(R) (2003).

[17] M. Mohseni, J. S. Lundeen, K. J. Resch, and A. M. Steinberg, Phys. Rev. Lett. **91**, 187903 (2003).

[18] M. S. Tame, R. Prevedel, M. Paternostro, P. Böhi, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **98**, 140501 (2007).

[19] D. Deutsch and R. Jozsa, *Proc. Roy. Soc. London Ser. A* **439**, 553 (1992).

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