Adv. Studies Theor. Phys., Vol. 4, 2010, no. 7, 317 - 322

A Type III Robinson-Trautman Metric Representing an Einstein-Maxwell Null Field

W. Davidson[1,2]

[1]Mathematics and Statistics Department
University of Otago, Dunedin, New Zealand

[2]Permanent Address: 21 Rowbank Way,
Loughborough, Leicestershire, UK, LE11 4AJ
wdav295@btinternet.com

Abstract. The equations are set up for a Robinson-Trautman Einstein-Maxwell field, employing a suitable complex null tetrad. A new Petrov Type III solution exhibiting an Einstein-Maxwell null field is presented. Modification of the solution then converts it to a Type III field of pure radiation.

Keywords: Exact, Robinson-Trautman, Einstein-Maxwell, Null

1. INTRODUCTION

Robinson –Trautman spacetimes [6], which in certain circumstances can be associated with radiating gravitational waves [6,2,5,3], are defined by the property that they admit a geodesic, shearfree, twistfree and diverging null congruence ***k.*** This means that the spin components of the congruence satisfy

$$\kappa = \sigma = 0, \ -\rho = -\bar{\rho} = \Theta > 0. \tag{1.1}$$

The metric for these spacetimes can be written in the form [6,7]:

$$\mathrm{d}s^2 = \frac{z^2}{P(t,x,y)^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - 2\mathrm{d}z\mathrm{d}t - 2N(t,x,y,z)\mathrm{d}t^2.$$

(1.2)

We label the coordinates *t,x,y* and *z* by x^i for $i = 0,1,2$ and 3, respectively.

It is convenient to refer the metric (1.2) to a complex null tetrad ***k, m, mbar*** and ***n,*** which in relation to the coordinate system x^i is written

$$
\begin{aligned}
k^i &= (0,\ 0,\ 0,\ 1),\\
m^i &= \left(0,\ P/(\sqrt{2}z),\ -\ iP/(\sqrt{2}z),\ 0\right),\\
\bar{m}^i &= \left(0,\ P/(\sqrt{2}z),\ iP/(\sqrt{2}z),\ 0\right),\\
n^i &= (1,\ 0,\ 0,\ -N).
\end{aligned}
\tag{1.3}
$$

We then find that certain tetrad components of the Ricci tensor vanish:

$$R_{00} = R_{11} = R_{22} = R_{10} = R_{20} = 0, \tag{1.4}$$

and for the invariant Θ we obtain

$$\Theta = 1/z. \tag{1.5}$$

Note that, for example, the tetrad Ricci component R_{00} in (1.4) is connected to the coordinate representation $R_{i\,j}$ via (1.3) by the relation

$$R_{00} = R_{i\,j} k^i k^j. \tag{1.6}$$

Examining the components of the Weyl tensor in the tetrad we derive

$$\psi_0 = \psi_1 = 0. \tag{1.7}$$

It follows that the metric (1.2) is *algebraically special.* Moreover we find that in the tetrad

$$R_{a[b} k_{c]} k^a = 0, \tag{1.8}$$

which indicates that the congruence ***k*** at any point is a Ricci eigen direction. Also, since $\psi_0 = 0$, ***k*** is also aligned as a principal null direction of the Weyl tensor.

2. THE EINSTEIN-MAXWELL ELECTROMAGNETIC FIELD

For an Einstein-Maxwell solution the Ricci scalar *R* must vanish. For the metric (1.2) this leads to the equation

$$R = \frac{2}{Pz^2}\left(KP - Pz^2 N_{zz} - 4PzN_z - 2PN - 6zP_t\right) = 0, \tag{2.1}$$

where a subscript indicates differentiation. Here *K* is proportional to the curvature of a 2-space *t* = const, *z* = const:

$$K(t,x,y) = P\nabla^2 P - P_x{}^2 - P_y{}^2,\quad \nabla^2 \equiv \partial^2{}_{x^2} + \partial^2{}_{y^2}. \tag{2.2}$$

We shall seek a Petrov Type III solution. Previous Type III solutions have been given by Bartrum [1] and by Ivanov [4]. Accordingly, in addition to (1.7) we must have

$$\psi_2 = \frac{1}{6Pz^2}\left(-KP + Pz^2 N_{zz} + PzN_z + 3zP_t + 2PN\right) = 0. \tag{2.3}$$

From (2.1) and (2.3) we derive the relation

$$PN_z + P_t = 0. \tag{2.4}$$

Integrating, we obtain for N:

$$N = -zP_t / P + Q(t,x,y). \tag{2.5}$$

Substituting in (2.1) there follows

$$Q = K/2, \tag{2.6}$$

and hence

$$N(t,x,y,z)) = K/2 - zP_t/P. \tag{2.7}$$

With the usual notation (see e.g. [7]), we now find for the Weyl invariants:

$$I = J = 0, \quad \psi_3 \neq 0, \tag{2.8}$$

which confirms that the metric is of Petrov Type III, having the form

$$\mathrm{d}s^2 = \left(z^2/P^2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - 2\mathrm{d}z\mathrm{d}t - \left(K - 2zP_t/P\right)\mathrm{d}t^2, \tag{2.9}$$

which has only $P(t,x,y)$ as unknown.

Examination of the tetrad components of the Ricci tensor shows that only R_{33} survives. In fact calculation yields

$$R_{ab} = \frac{P^2}{2z^2}\nabla^2 K \delta_a^3 \delta_b^3. \tag{2.10}$$

We therefore seek an electromagnetic field F_{ab} that leads to a field energy tensor T_{ab} for which only T_{33} is non-zero. The energy tensor of the field is

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{2.11}$$

where g_{ab} are the metric components with respect to the null tetrad:

$$g_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

This gives

$$T_{ab} = 2\delta_a^3\delta_b^3 F_{13}F_{23}, \tag{2.13}$$

provided we choose the only surviving components of F_{ab} to be

$$F_{13} = -F_{31} = f_1(t,x,y,z), \quad F_{23} = -F_{32} = f_2(t,x,y,z) \tag{2.14}$$

A consequence of this is that the invariants I_1 and I_2 vanish. That is,

$$I_1 = F_{ab}F^{ab} = 0, \quad I_2 = F_{ab}\tilde{F}^{ab} = 0, \tag{2.15}$$

where the dual bivector $\tilde{F}_{ab}$ is given by

$$\tilde{F}_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}, \tag{2.16}$$

ε_{abcd} being the Levi-Civita tensor in the complex null tetrad.

From (2.15) it follows that the electromagnetic field is *null*. In fact, in consequence of (2.14) the electromagnetic field invariants Φ_0, Φ_1 and Φ_2 have values

$$\Phi_0 = F_{ab}k^a m^b = 0,$$
$$\Phi_1 = \frac{1}{2}F_{ab}\left(k^a n^b + \bar{m}^a m^b\right) = 0, \qquad (2.17)$$
$$\Phi_2 = F_{ab}\bar{m}^a n^b = F_{23} = f_2.$$

Hence the discriminant $\Phi_0\Phi_2 - \Phi_1{}^2 = 0,$ confirming the *null* field.

We then note that we can write, in the tetrad:

$$T_{ab} = \Phi^2 k_a k_b, \qquad (2.18)$$

where

$$\Phi^2 = 2\Phi_2\bar{\Phi}_2 = 2 f_1 f_2. \qquad (2.19)$$

3. DETERMINATION OF THE FORM OF *f*1 AND *f2*

There are also two differential equations that F_{ab} must satisfy in an Einstein-Maxwell field:

$$F^{ab}_{;b} = 0, \qquad (3.1)$$

and

$$\tilde{F}^{ab}_{;b} = 0. \qquad (3.2)$$

Calculation of these equations relative to the complex null tetrad leads to the following forms:

$$f_1(t,x,y,z) = \frac{P}{2z}\left(b(t,x,y) - ic(t,x,y)\right) = \bar{\Phi}_2, \qquad (3.3)$$

$$f_2(t,x,y,z) = \frac{P}{2z}\left(b(t,x,y) + ic(t,x,y)\right) = \Phi_2, \qquad (3.4)$$

where b(t,x,y) and c(t,x,y) are real functions which (3.1) and (3.2) require to satisfy the relations

$$\frac{\partial b(t,x,y)}{\partial x} = -\frac{\partial c(t,x,y)}{\partial y},$$
$$\frac{\partial b(t,x,y)}{\partial y} = \frac{\partial c(t,x,y)}{\partial x}. \qquad (3.5)$$

The equations (3.5) show that the complex expression $b(t,x,y) + ic(t,x,y)$ is a function of $\bar{\zeta} = x - iy,$ and combining (2.10), (2.18), (2.19, (3.3) and (3.4) we can write

$$\left(b - ic\right)\left(b + ic\right) = b^2 + c^2 = \nabla^2 K. \qquad (3.6)$$

It therefore follows from (3.5) that a restriction on $P(t,x,y)$,

$$\nabla^2 \log\left(\nabla^2 K\right) = 0, \qquad (3.7)$$

is a necessary criterion for a valid solution.

We note in passing that in the original metric frame (2.9) the electromagnetic field has the non-zero values:

$$f(1,0) = -f(0,1) = b(t,x,y)/\sqrt{2},$$
$$f(2,0) = -f(0,2) = c(t,x,y)/\sqrt{2}. \qquad (3.8)$$

4. A SOLUTION

We choose the following value for $P(t,x,y)$:

$$P = a_0(t)e^{ax+by+m} + b_0(t)e^{cx+(a-b+c)y+n}, \qquad (4.1)$$

where a, b, c, m and n are constants with a+c>0 and $a_0(t) > 0, b_0(t) > 0$. This gives

$$\nabla^2 K = 4a_0 b_0 (a+c)^2\left((a-b)^2 + (c-b)^2\right)e^{(a+c)(x+y)+m+n}. \qquad (4.2)$$

We may therefore verify that (3.7) is satisfied. Taking account of (3.5) and (3.6) we determine appropriate values for $b(t,x,y)$ and $c(t,x,y)$:

$$b(t,x,y) = 2a_0^{1/2}b_0^{1/2}(a+c)\left((a-b)^2 + (c-b)^2\right)^{1/2} e^{\frac{1}{2}((a+c)(x+y)+m+n)} \cos\left\{\frac{1}{2}(a+c)(x-y)\right\}, \qquad (4.3)$$

$$c(t,x,y) = 2a_0^{1/2}b_0^{1/2}(a+c)\left((a-b)^2 + (c-b)^2\right)^{1/2} e^{\frac{1}{2}((a+c)(x+y)+m+n)} \sin\left\{\frac{1}{2}(a+c)(x-y)\right\}. \qquad (4.4)$$

$f_1(t,x,y,z)$ and $f_2(t,x,y,z)$, as well as Φ_2 and $\bar{\Phi}_2$, are then given by (3.3) and (3.4). Thus we obtain for Φ^2:

$$\Phi^2 = \frac{2P(t,x,y)^2}{z^2} a_0(t)b_0(t)(a+c)^2\left((a-b)^2 + (c-b)^2\right)e^{(a+c)(x+y)+m+n}, \qquad (4.5)$$

and in the original metric frame (2.9) the components of F_{ab} are given by (3.8).

We have therefore derived a Robinson-Trautman spacetime exhibiting a Petrov type III Einstein-Maxwell null field.

5. A TYPE III PURE RADIATION FIELD

In the case of *pure radiation* the energy tensor of the field has the construction, in the tetrad:

$$T_{ab} = \Phi^2 k_a k_b. \qquad (5.1)$$

In a Robinson-Trautman spacetime, because of (1.1), Φ^2 has to take the form (see e.g. [7]):

$$\Phi^2 = q(t,x,y)/z^2, \quad q(t,x,y) > 0. \qquad (5.2)$$

Assuming $R = 0$, and seeking a Type III solution $(\psi_0 = \psi_1 = \psi_2 = 0)$, we can adopt the metric (2.9). It follows that (2.10) applies for $T_{ab} (= R_{ab})$. Hence we can write

$$q(t, x, y) = \frac{1}{2} P^2 \nabla^2 K. \tag{5.3}$$

If we now choose (4.1) for the value of $P(t, x, y)$ then, by (4.2),

$$q(t, x.y) = 2a_0(t) b_0(t)(a+c)^2 \left((a-b)^2 + (c-b)^2\right) e^{(a+c)(x+y)+m+n} P(t, x, y)^2, \tag{5.4}$$

which is > 0, as required.

Equations (5.1)-(5.4) then provide a Robinson-Trautman Type III spacetime evincing a pure radiation solution.

REFERENCES

[1] P.C. Bartrum, J. Math. Phys., 8 (1967),1464.

[2] P.T. Chrusciel, Commun. Math. Phys., 137 (1991), 289.

[3] C. Hoenselaers and Z. Perzes, Class. Quantum Grav., 10 (1993), 375.

[4] G.G. Ivanov, Grav. Teor. Otnos., 12 (1977), 69.

[5] B. Lukacs, Z. Perzes, J. Porter and A. Sebestyen, Gen. Rel. Grav., 16 (1984), 691.

[6] I. Robinson and A. Trautman, Proc. Roy. Soc. Lond., A 265 (1962), 463.

[7] H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge (2003).

Received: February, 2010