Adv. Studies Theor. Phys., Vol. 4, 2010, no. 8, 383 - 392

$\mathbb{Z}_3$-Graded Geometric Algebra

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Abstract

By considering the $\mathbb{Z}_2$ gradation structure, the aim of this paper is constructing $\mathbb{Z}_3$ gradation structure of multivectors in geometric algebra.

Mathematics Subject Classification: 16E45, 15A66

Keywords: Graded Algebra, Geometric Algebra

1 Introduction

The foundations of geometric algebra, or what today is more commonly known as Clifford algebra, were put forward already in 1844 by Grassmann. He introduced vectors, scalar products and extensive quantities such as exterior products. His ideas were far ahead of his time and formulated in an abstract and rather philosophical form which was hard to follow for contemporary mathematicians. Because of this, his work was largely ignored until around 1876, when Clifford took up Grassmann's ideas and formulated a natural algebra on vectors with combined interior and exterior products. He referred to this as an application of Grassmann's geometric algebra.

Due to unfortunate historic events, such as Clifford's early death in 1879, his ideas did not reach the wider part of the mathematics community. Hamilton had independently invented the quaternion algebra which was a special case of Grassmann's constructions, a fact Hamilton quickly realized himself. Gibbs reformulated, largely due to a misinterpretation, the quaternion algebra to a system for calculating with vectors in three dimensions with scalar and cross products. This system, which today is taught at an elementary academic level, found immediate applications in physics, which at that time circled around Newton's mechanics and Maxwell's electrodynamics. Clifford's algebra only continued to be employed within small mathematical circles, while physicists struggled to transfer the three-dimensional concepts in Gibbs' formulation to special relativity and quantum mechanics. Contributions and independent reinventions of Grassmann's and Clifford's constructions were made along the way by Cartan, Lipschitz, Chevalley, Riesz, Atiyah, Bott, Shapiro, and others. Only in 1966 did Hestenes identify the Dirac algebra, which had been constructed for relativistic quantum mechanics, as the geometric algebra of spacetime. This spawned new interest in geometric algebra, and led, though with a certain reluctance in the scientific community, to applications and reformulations in a wide range of fields in mathematics and physics. More recent applications include image analysis, computer vision, robotic control and electromagnetic field simulations. Geometric algebra is even finding its way into the computer game industry.[4]

In this paper we consider to graded structute of geometric algebra and then formalize it to construct $\mathbb{Z}_3$-graded structure. Also we use this structure on two important geometric algebras: Grassmann and Clifford.

This paper is organized as follows: second section contains the core structure of graded spaces and consider to $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gradation. In the next section we give definition of graded algebras. At last we use gradation structures to Grassmann and Clifford algebras.

2 Graded Algebras

In this section we give some preliminary definitions and results about Graded Algebras.

Definition 2.1 *Let G be a finite additive abelian group and $\mathbb{F}$ be the real or complex field. A map $\sigma : G \times G \longrightarrow \mathbb{F}$ is called the* **sign** *or* **commutation factor** *of G if it satisfies*

- $\sigma(\alpha, \beta)\sigma(\beta, \alpha) = 1$
- $\sigma(\alpha + \beta, \gamma) = \sigma(\alpha, \beta)\sigma(\beta, \gamma)$

for any $\alpha, \beta, \gamma \in G$. The pair (G, σ) is called **signed group**.

It is easy to verify that $\sigma(\alpha,\alpha) = \pm 1$ for any $\alpha \in G$. An element α of G is called **even** (resp. **odd**) if $\sigma(\alpha,\alpha) = 1$(resp. -1). The even part $\{\alpha \in G : \sigma(\alpha,\alpha) = 1\}$ and the odd part $\{\alpha \in G : \sigma(\alpha,\alpha) = -1\}$ of G are denoted by G_0 and G_1 respectively. G_0 is a subgroup of G of index at most 2 and we have $G = G_0 \cup G_1$(disjoint union).

Definition 2.2 *A mapping $\phi : G \times G \longrightarrow \mathbb{F} - \{0\}$ is called a* **factor system** *on G, if for any $\alpha, \beta, \gamma \in G$, it satisfies*

(1) $\phi(\alpha, \beta+\gamma)\phi(\beta,\gamma) = \phi(\alpha,\beta)\phi(\alpha+\beta,\gamma)$;

(2) $\phi(0,0) = 1$,

It follows from (1) that

* $\phi(\alpha, 0) = \phi(0, \alpha) = 1$;
* $\phi(\alpha, -\alpha) = \phi(-\alpha, \alpha) = \phi(\alpha,\beta)\phi(-\alpha, \alpha+\beta)$;

for any $\alpha, \beta \in G$.

Proposition 2.3 *Let (G, σ) be an even signed group and assume that G is finitely generated. Then there is a factor system ϕ on G such that $\sigma(\alpha,\beta) = \phi(\alpha,\beta)/\phi(\beta,\alpha)$ for $\alpha, \beta \in G$. Moreover, if $|\sigma(\alpha,\beta)| = 1$ for all $\alpha, \beta \in G$, we can choose ϕ so that $|\phi(\alpha,\beta)| = 1$ for all $\alpha, \beta \in G$.*

Definition 2.4 *A vector space V is said to be G-***graded** *if we are given a family $(V_\alpha)_{\alpha \in G}$ of subspaces of V such that V is their direct sum, $V = \bigoplus_{\alpha \in G} V_\alpha$.*

An element of V is said to be **homogeneous of grade** $\alpha \in G$ if it is an element of V_α. Let V and W be two G-graded vector spaces. A linear mapping $T : V \to W$ is said to be **homogeneous of grade** $\alpha \in G$ if $T(V_\beta) \subset W_{\alpha+\beta}$ for all $\beta \in G$.

Let $L(V, W)$ denote the vector space of all linear mappings of V into W and let $L_\alpha(V, W)$ denote the subspace of those linear mappings of V into W which are homogeneous of grade α. We define $L_{gr}(V, W)$ to be the sum of these subspaces, obviously this sum is directed:

$$L_{gr}(V, W) = \bigoplus_{\alpha \in G} L_\alpha(V, W).$$

Thus $L_{gr}(V, W)$ is a G-graded vector space. Note that $L_{gr}(V, W)$ is equal to $L(V, W)$ if (for example) $V_\alpha = \{0\}$ and $W_\alpha = \{0\}$ for all but a finite number of degrees. In the case where $V = W$ and $V_\alpha = W_\alpha$ for all $\alpha \in G$, we shall simplify the notations and write $L(V)$ and $L_{gr}(V)$ instead of $L(V,V)$ and $L_{gr}(V,V)$, respectively.

Let U, V and W be three G-graded vector spaces and let $h : U \to V$ and $k : V \to W$ be two linear mappings. If h is homogeneous of grade α and k is homogeneous of grade β, then koh is homogeneous of degree $\alpha + \beta$.

Definition 2.5 *An algebra $\mathcal{A}$ is called* G**-graded algebra** *if $\mathcal{A}$ has direct sum decomposition $\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha$ where $\mathcal{A}_\alpha$ is a subalgebra of $\mathcal{A}$ of grade α for any $\alpha \in G$ with additional condition that $\mathcal{A}_\alpha \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}$ for all $\alpha, \beta \in G$.*

A G-graded (associative) algebra $\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha$ over $\mathbb{F}$ is called σ**-commutative** *algebra if $ab = \sigma(\alpha, \beta)ba$ holds for any $a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta$ and $\alpha, \beta \in G$.*

It is important to note that in the case char $\mathbb{F}$=2 we must add the condition $a^2 = \sigma(\alpha, \alpha)a^2$ for any $a \in \mathcal{A}_\alpha$.

Definition 2.6 Graded tensor product
For two G-graded algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{F}$, the G-graded vector space $\mathcal{A} \otimes \mathcal{B} = \bigoplus_{\alpha \in G} (\bigoplus_{\beta+\gamma=\alpha} (\mathcal{A}_\beta \otimes \mathcal{B}_\gamma))$ is a G-graded algebra if, we define the multiplication by $(a \otimes b).(c \otimes d) = \sigma(\beta, \gamma)(ac \otimes bd)$ for $\beta, \gamma \in G$ and $a \in \mathcal{A},\ b \in \mathcal{B}_\beta,\ c \in \mathcal{A}_\gamma$ and $d \in \mathcal{B}$. The algebra $\mathcal{A} \otimes \mathcal{B}$ is called the **graded tensor product** *of $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{F}$. If $\mathcal{A}$ and $\mathcal{B}$ are σ-commutative, so is $\mathcal{A} \otimes \mathcal{B}$.*

Let $V = \bigoplus_{\alpha \in G} V_\alpha$ be a G-graded vector space over $\mathbb{F}$. The G-graded vector space

$$V \otimes V = \bigoplus_{\alpha \in G} \Big(\bigoplus_{\beta+\gamma=\alpha} (V_\beta \otimes V_\gamma) \Big)$$

is a G-graded algebra if, we define the multiplication by

$$(u \otimes v).(w \otimes s) = \sigma(\beta, \gamma)(uw \otimes vs)$$

for $\beta, \gamma \in G$ and $u, s \in V,\ v \in V_\beta, w \in V_\gamma$.

For any $n \in \mathbb{Z}$, let

$$T_n(V) = \underbrace{V \otimes V \otimes V \cdots \otimes V}_{n \text{ times}} = \bigotimes_n V$$

$T_0(V) = \mathbb{F}$, $T_1(V) = V$ and $T_n(V) = \{0\}$ for $n \leq -1$.

Definition 2.7 ***Tensor Algebra***
For a G-graded vector space, tensor algebra of V over $\mathbb{F}$ is defined as

$$T(V) = \bigoplus_{n=0}^{\infty} T_n(V)$$

As is well-known, $T(V)$ has a natural $\mathbb{Z}\times G$-gradation which is fixed by the condition that the grade of a tensor $v_1 \otimes \cdots \otimes v_n$, with $v_i \in V_{\alpha_i}$, $\alpha_i \in G$ for $1 \leq i \leq n$, is equal to $(n, \alpha_1, \ldots, \alpha_n)$. $T_n(V)$ is a subspace of $T(V)$ consisting of the homogeneous tensors of order $n \in \mathbb{Z}$.

2.1 $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gradation

Now we consider to the most important groups, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ which are abelian. To construct gradation using these groups, we have to define sign or commutation factor on them.

$\mathbb{Z}_2$ gradation:

Let $\mathbb{F}$ be complex number field. A mapping $\sigma : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{F}$ defined by

$$\sigma(\alpha,\beta) = (-1)^{\alpha\beta}, \quad \text{for all} \quad \alpha,\beta \in \mathbb{Z}_2 \tag{1}$$

is a sign on $\mathbb{Z}_2$ which satisfies conditions of definition 2.1. In this case a vector space V is called Z_2-graded if it can be decomposed to direct sum of two subspace as $V = V_0 \oplus V_1$. Also for a Z_2-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, the multiplication is said to be *anticommutative* if it satisfies

$$ab = (-1)^{\alpha\beta} ba, \quad \text{for all} \quad a \in \mathcal{A}_\alpha \quad \text{and} \quad b \in \mathcal{A}_\beta, \quad \alpha,\beta \in \mathbb{Z}_2$$

$\mathbb{Z}_3$ gradation:

Consider the cyclic group $\mathbb{Z}_3$. It can be represented in the complex plane as multiplication by primary cubic roots of unity $q = e^{\frac{2\pi i}{3}} (i = \sqrt{-1})$

$$q^3 = 1, \quad \text{and} \quad q^2 + q + 1 = 0 \quad .$$

A mapping $\sigma : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{F}$ defined by

$$\sigma(\alpha,\beta) = q^{\alpha\beta}, \quad \text{for all} \quad \alpha,\beta \in \mathbb{Z}_3 \tag{2}$$

is called *sign* (or *comutation factor*) of $\mathbb{Z}_3$ which also satisfies the conditions of definition 2.1.

Let V be an n dimensional vector space which is $\mathbb{Z}_3$-graded over $\mathbb{F}$, this means that we can write V as direct sum of three sets, $V = V_0 \oplus V_1 \oplus V_2$.

Also for a Z_3-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2$, the multiplication is said to be *ternary anticommutative* if it satisfies

$$ab = q^{\alpha\beta} ba, \quad \text{for all} \quad a \in \mathcal{A}_\alpha \quad \text{and} \quad b \in \mathcal{A}_\beta, \quad \alpha,\beta \in \mathbb{Z}_3$$

3 Geometric Algebra

This section is devoted to construct Z_3-graded geometric algebra according to the previous concepts.

For an n dimensional $\mathbb{Z}_3$-graded vector space V over $\mathbb{F}$, the $\mathbb{Z}_3$-graded vector space

$$V \otimes V = \bigoplus_{\alpha \in \mathbb{Z}_3} \Big(\bigoplus_{\beta+\gamma=\alpha} (V_\beta \otimes V_\gamma) \Big)$$

is a $\mathbb{Z}_3$-graded algebra if, we define the multiplication by

$$(u \otimes v).(w \otimes s) = \sigma(\beta, \gamma)(uw \otimes vs)$$

for $\beta, \gamma \in \mathbb{Z}_3$ and $u, s \in V$, $v \in V_\beta, w \in V_\gamma$. The algebra $V \otimes V$ is σ-commutative, which is called $\mathbb{Z}_3$-graded tensor product. For every two elements $u, v \in V$, which $u = u_1 \oplus u_2 \oplus u_3$ and $v = v_1 \oplus v_2 \oplus v_3$, the element of $V \bigotimes V$ has the explicit form:

$$u \otimes v = \{u_0 \otimes v_0\} \oplus \{u_0 \otimes v_1 \oplus u_1 \otimes v_0\} \oplus \{u_0 \otimes v_2 \oplus u_1 \otimes v_1 \oplus u_2 \otimes v_0\}$$

In this manner we can see that for $k \in \mathbb{Z}^+$, the set $\bigotimes^k V$ of k-tensors is a $\mathbb{Z}_3$-graded σ-commutative algebra. We define $\bigotimes^0 V = FF$ and $\bigotimes^1 V = V$. The elements of this space will be called k-vectors.
Let

$$T(V) = \bigoplus_{k=0}^{\infty} \bigotimes^k V \tag{3}$$

be the tensor algebra of V over $\mathbb{F}$, the elements of which are finite sums of tensors of arbitrary finite grades on V. As is well-known, $T(V)$ has a natural $\mathbb{Z} \times \mathbb{Z}_3$-gradation which is fixed by the condition that the grade of a tensor $v_1 \otimes \cdots \otimes v_k$, with $v_i \in V_{\alpha_i}, \alpha_i \in \mathbb{Z}_3$ for $1 \leq i \leq k$, is equal to $(n, \alpha_1, \cdots, \alpha_k)$.

Consider the two-sided ideal generated by all elements of the form $v_\alpha v_\beta - \sigma(\alpha, \beta) v_\beta v_\alpha$ where $\alpha, \beta \in \mathbb{Z}_3$ and $v_\alpha \in V_\alpha, v_\beta \in V_\beta$.

This means that

$$I(V) := \Big\{ \sum_k A_k \otimes \big(v_\alpha v_\beta - \sigma(\alpha, \beta) v_\beta v_\alpha\big) \otimes B_k, v_\alpha \in V_\alpha, v_\beta \in V_\beta,\ A_k, B_k \in T(V) \Big\}$$

We define the geometric algebra over V by quoting out this ideal from $T(V)$.

Definition 3.1 *The* geometric algebra $GA(V)$ *over the vector space* V *is defined by*

$$GA(V) := T(V)/I(V).$$

This quotient algebra, is a σ-commutative Z_3-graded algebra.

When it is clear from the context what vector space we are working with, we will often denote by G.

The product in GA, called the *geometric* or *Clifford product*, is inherited from the tensor product in $T(V)$ and we denote it by juxtaposition (or $\cdot$ if absolutely necessary),

$$\begin{aligned} GA \times GA &\to GA, \\ (A, B) &\mapsto AB := [A \otimes B] = A \otimes B + I. \end{aligned}$$

Note that this product is bilinear and associative.

Geometric Algebra is a general case for two important algebras, Grassmann and clifford algebra. Here we consider there structure as special cases of geometric algebra.

Consider to 3. The subspace of $T(V)$ consisting of the homogeneous tensors of order $k \in \mathbb{Z}$ will be deoted by $T_n(V)$; indeed

$$T_n(V) = \bigoplus_{k=0}^{n} \bigotimes^k V$$

of course, $T_n(V) = \{0\}$ if $n \leq -1$. $T_0(V) = \mathbb{F}$ and $T_1(V) = V$.

This subspace is the 2^n dimensional vector space of multivectors over V. When equeped with the exterior product $\wedge$, the vector space $T_n(V) = \Lambda_n(V)$ is called *Grassmann algebra* over V. Note that $\Lambda_n(V)$ is another example of a $\mathbb{Z}\times\mathbb{Z}_3$-gradation, with a $\mathbb{Z}$-graded structure inherited from the usual $\mathbb{Z}$-grading of $T(V)$.

4 graded Graßmann Algebra

In this section we consider to an important algebra which has wide usage in theoretical physics and Mathematics. Of course physicists and mathematicians use this algebra instead of number fields. So it's elements is also called **supernumbers**. Here, we consider to structure of this algebra in details.

It is important to note that a Grassmann algebra is a geomeric algebra with special multiplication which is called outer product.

$\mathbb{Z}_2$ Graßmann algebra

The **Graßmann algebra**(or **exterior algebra**)Λ_n with n generators is the associative algebra (over $\mathbb{C}$) generated by a set of n anticommuting generators $\{\xi_i\}_{i=1}^n$ and by $1 \in \mathbb{C}$ with the property

$$\xi_i\xi_j = -\xi_j\xi_i \qquad \text{for all } i, j, \tag{4}$$

- The second power (or higher) of any generator vanishes :

$$(\xi_i)^2 = 0 \tag{5}$$

- Any product of three or more generators also vanishes :

$$\xi_i \xi_j \xi_k = 0 \qquad i, j, k = 0, 1, ..., N \tag{6}$$

It follows from 4 that any element of Λ_n is linear combination of monomials $\xi_{m_1}\xi_{m_2}...\xi_{m_k}$ with $1 \leq m_1 < m_2 < \cdots < m_k \leq n$ and the unit such that $1 \leq k \leq n$. Since all monomials among ξ_i follow from 4, these monomials are linearly independent. Consequently, together with the unit, they form a basis of Λ_n as a linear space. Since their number is equal to the number of subsets of n elements, we have dim $\Lambda_n = 2^n$.

Any element $\lambda \in \Lambda_n$ may be written as

$$\lambda = \lambda(\xi) = \sum_{k \geq 0} \sum_{m_1, \ldots, m_k} \lambda m_1, \ldots, m_k \xi_{m_1}\xi_{m_2}...\xi_{m_k}. \tag{7}$$

The term corresponding to $k = 0$ is proportional to the unit. The relation $\lambda = \lambda(\xi)$ shows the fact that λ is expressed in the form of a polynomial in ξ_m. The expression in elements of Λ_n in the above form is not unique in general. This becomes unique if supplementary conditions are imposed on coefficients $\lambda m_1, \ldots, m_k$. For instance, we may require that $\lambda m_1, \ldots, m_k = 0$ whenever the relation $m_1 < m_2 < \cdots < m_k$ fails. Let

$$M_n = \{(m_1, \ldots, m_k)| \ ; 1 \leq k \leq n \ ; 1 \leq m_1 < \cdots < m_k \leq n \ n, m_i \in \mathbb{N}\}. \tag{8}$$

Therefore with this condition any element $\lambda \in \Lambda_n$ can be written uniquely as

$$\begin{aligned} \lambda &= \sum_{k \geq 0} \sum_{\substack{m_1, \ldots, m_k \\ m_1 < m_2 < \cdots < m_k}} \lambda m_1, \ldots, m_k \xi_{m_1}\xi_{m_2}...\xi_{m_k} \\ &= \lambda_0 + \sum_{(m_1, \ldots, m_k) \in M_n} \lambda m_1, \ldots, m_k \xi_{m_1}\xi_{m_2}...\xi_{m_k} \end{aligned} \tag{9}$$

such that $\lambda_0 \in \mathbb{C}$ is the number for $k = 0$.

$\mathbb{Z}_3$-graded Grassmann algebra

$\mathbb{Z}_3$ is the cyclic group of three elements. It can be represented in the complex plane as multiplication by the primary cubic root of unity $q = e^{\frac{2\pi i}{3}}$, q^2 and $q^3 = 1$. The analog of the Z_2-graded Grassmann algebra can be introduced as follows:

Consider an associative algebra spanned by N generators ξ_i; $\quad i = 0, = 1, ..., N$, between which only ternary relations exist. This means that the binary products of any two of such elements are considered as independent entities, i.e. $\xi_i \xi_j$ are independent of $\xi_j \xi_i$ where $i, j = 0, 1..., N$). In this case the ternary commutation is given by the following ternary relations:

$$\xi_i \xi_j \xi_k = q \xi_j \xi_k \xi_i = q^2 \xi_k \xi_i \xi_j \qquad i, j, k = 0, 1, ..., N \tag{10}$$

Two important properties follow automatically:

- The third power (or higher) of any generator vanishes :

$$(\xi_i)^3 = 0 \tag{11}$$

- Any product of four or more generators also vanishes :

$$\xi_i \xi_j \xi_k \xi_l = 0 \qquad i, j, k, l = 0, 1, ..., N \tag{12}$$

we can associate grade 0 to the identity element and grade 1 to the generators ξ's. By defining $\bar{\xi} = \xi^2$, for every generator ξ, we can see that

- Grade-0 : $I, \;\; \xi\bar{\xi}, \;\; \xi\xi\xi, \;\; \bar{\xi}\bar{\xi}\bar{\xi}$
- Grade-1 : $\xi, \;\; \bar{\xi}\bar{\xi}$
- Grade-2 : $\bar{\xi}, \;\; \xi\xi$

The dimension of resulting algebra is: $D = \dfrac{3 + 4N + 9N^2 + 2N^3}{3}$

5 Graded Clifford Algebra

Let V be an n-dimensional real vector space and $T(V) = \bigoplus_{k=0}^{\infty} \bigotimes^k V$ be the tensor algebra over V. Denote the space of antisymmetric k-tensors by $\bigotimes^k V$, which its elements are called k-vectors. Let $T_n(V) = \bigoplus_{k=0}^{n} \bigotimes^k V$ denote the 2^n-dimensional real vector space of multivectors over V.

The clifford product between a vector $v \in V$ and a multivector a in $T_n(V)$ is given by $va = v \wedge a + v.a$. The resulting algebra is the so called **Clifford algebra** $\mathcal{C}\ell(V)$.

According to [5], the usual $\mathbb{Z}_2$- grading of $\mathcal{C}\ell(V)$ is given by $\mathcal{C}\ell^+(V) \oplus \mathcal{C}\ell^-(V) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$ where $\mathcal{C}\ell_0 = \bigoplus_{k \text{ even}} \bigotimes^k V$ and $\mathcal{C}\ell_1 = \bigoplus_{k \text{ odd}} \bigotimes^k V$. In this way $\mathcal{C}\ell(V)$ is given by a direct sum of subspaces $\mathcal{C}\ell_i, i = 0, 1$ which satisfy $\mathcal{C}\ell_i \mathcal{C}\ell_j \subseteq \mathcal{C}\ell_{i+j(\text{mod } 2)}$.

Also $\mathbb{Z}_3$-grading of $\mathcal{C}\ell(V)$ is given by $\mathcal{C}\ell(V, g) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1 \oplus \mathcal{C}\ell_2$ where $\mathcal{C}\ell_0 = \bigoplus_{3k} \bigotimes^k V$, $\mathcal{C}\ell_1 = \bigoplus_{3k+} \bigotimes^k V$ and $\mathcal{C}\ell_2 = \bigoplus_{3k+2} \bigotimes^k V$.

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Received: January, 2010