

# Parametric Excitation of an Electrostatic Wave in a Nonuniform Relativistic Warm Plasma Waveguide

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## Abstract

The propagation of an electrostatic wave in a  $2 - D$  nonuniform relativistic warm plasma waveguide under the effect of HF electric field is investigated. A new mathematical technique “separation method” applied to the two-fluid plasma model to separate the equations, which describe the system, into two parts time and space parts. An analytical study of the reflection of electrostatic wave propagation along a magnetized nonuniform relativistic warm plasma slab subjected to an intense HF electric field is presented and compared with the non relativistic cold plasma case. It is found that when the frequency of the incident wave  $\omega_i$  is close to the relativistic warm electron plasma frequency  $\omega_{pe}$ , the plasma is less reflective due to presence of the HF field and the effect of relativistic warm electrons. On the other hand, for a low-frequency incident wave ( $\omega_i \sim \omega_{pi}$ ) the reflection coefficient is directly proportional to the amplitude of the HF field. Also, it is shown that the relativistic warm electrons plasma lead to a decrease of the value of the reflection coefficient in comparison with the case of non relativistic cold plasma.

**Keywords:** Propagation of electrostatic wave, Relativistic warm plasma waveguide, the reflection coefficient, Separation method.

## 1. Introduction

Parametric instability plays an important role from the point of view of heating plasma by absorbing energy from external applied-frequency electromagnetic fields which provide the deriving force. Although in the case when plasma electrons are

responsible for parametric phenomena, positive ions, as a background, strongly affect that phenomenon. This subject showed to be a great interest and have been considered by many authors.

The separation method are used for the solution of different problems: (1) the stabilization effect of a strong HF electric on a two-stream (Buneman) instability in a plane and cylindrical plasma waveguides has been discussed in [1-3], (2) study the effect of spatial plasma nonuniformity on parametric instability of electrostatic wave in a plasma waveguides subjected to an intense HF electric field has been performed in [4, 5], and (3) an analytical expression of the reflection coefficient for the electrostatic wave propagating along a nonuniform cold plasma slab immersed into high-amplitude HF electric field has been presented in [6].

The parametric interaction of an external HF electric field with an electrostatic surface wave in isotropic nonuniform plasma has been previously investigated using a special method based on the separation of variables [7]. The method makes it possible to separate the problem into two parts. The “dynamical” (temporal) part describes the parametric excitation of waves and corresponding equations within the renormalization of natural (eigen) frequencies coincide with equations for parametrically unstable waves in an uniform plasma [8,9]. Natural frequencies of surface waves and spatial distribution of the self-consistent electric field amplitude are determined from the solution of a boundary-value problem (“spatial” part) taking into account specific spatial distribution of plasma density. The proposed approach (“separation method”) is significantly simpler than the method previously used in the theory of parametric resonance in a nonuniform plasma (e.g., ref. [10] and references therein).

In ref. [7] the problem of parametric excitation of natural modes of semi-infinite plasma (surface waves) was analyzed as an initial value problem. In other words, surface waves are excited due to an initial perturbation at the boundary and a dispersion equation determines the complex frequency  $\omega$  as a function of the real wave number  $k$ . It is of practical interest to use the separation method described in [7] for the solution of an eigen-values problems when the wave number  $k$  is found as a function of the real frequency  $\omega$ . This means that one has to treat a forced oscillations excited in a plasma by an external source (generator) with a fixed frequency  $\omega_a$  (for more detail see e.g., [11], where the initial value problem for the problem of surface wave transformation at the plasma resonance in a transition layer).

The character of high- frequency field interaction with plasma is being widely discussed. This interest is due, first of all, to a partial or total high- frequency stabilization of various dangerous instabilities preventing prolonged plasma confinement in magnetic traps (see refs. [12-14] and the references quoted therein). Furthermore, under certain conditions, a strong high- frequency field causes parametric instabilities, when the electron and ion oscillation amplitudes (hence, their kinetic energy) increases appreciably (e.g., refs. [15, 16]). Since the temperature of

plasma particles increases considerably (e.g., [17]), hence, the high-frequency fields may “heat” the plasma up to rather high temperatures.

Here we shall analyze propagation of an electrostatic wave in a  $2-D$  nonuniform relativistic warm plasma waveguide subjected to an intense HF electric field as an eigen-value problem.

## 2. Geometry of the problem

Let us suppose that plane waveguide is filled by nonuniform plasma ( $n_{\alpha_n} = n_{\alpha_n}(x)$ ;  $\alpha = e, i$ ). A uniform strong static magnetic field  $\vec{B}_0$  ( $\omega_{ce} \gg \omega_{pe}$ ) and a HF electric field  $\vec{E}_p = \vec{E}_0 \sin(\omega_0 t)$  are directed along the  $z$  axis. We choose the electric field of an ordinary wave as an HF pump field. For the frequency of the ordinary waves we have  $\omega_0^2 = \omega_{pe}^2 + c^2 k_0^2$ , so that the pump electric field can be considered to be uniform if its frequency  $\omega_0 \approx \omega_{pe}$ . This is because the wavelength of the pump electric field must be larger than the characteristic scale of nonuniformity  $L$ ,  $(2\pi/k_0) \gg L$ . We assume also that plasma is bounded by an ideally conducting surfaces placed at  $y = \pm d$  and that plasma is uniform in the  $y$ -direction. The distribution of plasma density is supposed to have the form of the Epstein transition layer (for more details see [18, 19]):

$$\frac{n(x)}{N} = 1 - \frac{\mu}{1 + e^{x/L}} \quad (1)$$

Where  $\mu$  is constant. The geometry of the problem is presented in Figs. (1) and (2).

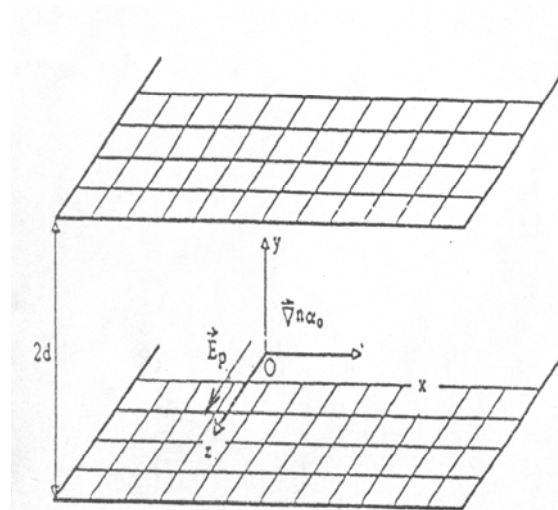


Fig. 1. The geometry of the problem on reflection of an electrostatic wave from the density gradient in a plane plasma waveguide.

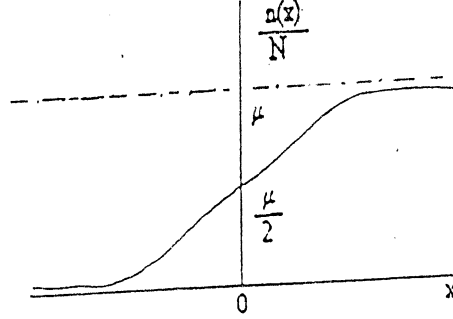


Fig. 2. "Axial" Profile of the equilibrium plasma density.

### 3. Separation method in the problem of electrostatic wave propagation in nonuniform relativistic plasma waveguide subjected to an intense HF electric field of arbitrary amplitude

The equilibrium particles velocity  $\vec{u}_\alpha(0,0,u_\alpha)$  is determined by the following expression:

$$\vec{u}_\alpha = -\frac{e_\alpha \vec{E}_0}{m_\alpha \omega_n} \cos(\omega_0 t) \quad (2)$$

Representing the perturbations of velocity  $\delta \vec{V}_\alpha(0,0,\delta V_\alpha)$ , density  $\delta n_\alpha$  and electrical potential  $\Phi$  in the form  $(\delta \vec{V}_\alpha, \delta n_\alpha, \Phi) \sim \exp(ikz)$ , the linearized equations of motion and continuity reads:

$$\frac{\partial^2 v_\alpha}{\partial t^2} - i\Delta_\gamma \frac{\partial v_\alpha}{\partial t} + \eta_\gamma v_\alpha = -k^2 n_{\alpha_0}(x,y) \frac{e_\alpha^2}{m_\alpha} e^{-iA_\alpha} \Phi(x,y) \quad (3)$$

Where;

$$v_\alpha = e_\alpha \delta n_\alpha e^{iA_\alpha}, \quad \Delta_\gamma = k u_0 (1 - \gamma), \quad \gamma = (1 - u_0^2/c^2)^{-1/2}, \quad A_\alpha = -a_\alpha \sin(\omega_0 t)$$

$$a_\alpha = \frac{e_\alpha k E_0}{m_\alpha \omega_0^2} \approx a_e, \quad \eta_\gamma = \gamma k^2 V_{th}^2.$$

The Poisson's equation takes the form

$$-\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} + k^2 \Phi = 4\pi \sum_\alpha v_\alpha e^{iA_\alpha} \quad (4)$$

Assuming  $v_\alpha(x, y, t) = v_{\alpha_1}(t)v_{\alpha_2}(x, y)$ ,  $\Phi(x, y, t) = \Phi_1(t)\Phi_2(x, y)$  and separating variables in equations (3) and (4), we find:

$$\frac{v_{e_2}}{k^2 \frac{e^2}{m_e} n_0 \Phi_2} = - \frac{\Phi_1 e^{-iA_e}}{\left[ \frac{\partial^2 v_{e_1}}{\partial t^2} - i \Delta_y \frac{\partial v_{e_1}}{\partial t} \right]} = \alpha_e, \quad (5)$$

$$\frac{v_{i_2}}{k^2 \frac{e^2}{m_i} n_0 \Phi_2} = - \frac{\Phi_1 e^{-iA_i}}{\frac{\partial^2 v_{i_1}}{\partial t^2}} = \alpha_i, \quad (6)$$

Where  $\alpha_e$  and  $\alpha_i$  are the separation constants;  $n_0 = n_{\alpha_n}(x)$  is a nonuniform plasma density and the relation between  $n_0(x)$  and the symbols  $\mu$  and  $N$  is shown from equation (1). From equations (5) and (6), we have:

$$v_{e_2} = \frac{\alpha_e m_i}{\alpha_i m_e} v_{i_2}, \quad v_{i_2} = \alpha_i k^2 \frac{e^2}{m_i} n_0 \Phi_2 \quad (7)$$

Eliminating by expressions (7), the quantities  $v_{e_2}$  and  $v_{i_2}$  from equation (4), the latter can be re-expressed in the form:

$$\frac{\frac{\partial^2 \Phi_2}{\partial x^2} - \frac{\partial^2 \Phi_2}{\partial y^2} + k^2 \Phi_2}{v_{i_2}} = \frac{4\pi \left( v_{e_1} \frac{\alpha_e m_i}{\alpha_i m_e} e^{iA_e} + v_{i_1} e^{iA_i} \right)}{\Phi_1} = \alpha_s, \quad (8)$$

Where  $\alpha_s$  is a separation constant. From the first part of equation (8), we obtain:

$$-\frac{\partial^2 \Phi_2}{\partial x^2} - \frac{\partial^2 \Phi_2}{\partial y^2} + k^2 \Phi_2 = \alpha_s v_{i_2} = \alpha_s \alpha_i k^2 \frac{e^2}{m_i} n_0 \Phi_2 \quad (9)$$

Because the quantities  $\alpha_s$  and  $\alpha_i$  are arbitrary constants we can choose them in the following way:

$$\alpha_s \alpha_i = + \frac{m_i}{m_e} \frac{4\pi}{p^2} \quad (10)$$

Where  $p$  is a constant value. Using (10), we find from (9):

$$\frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} - k^2 \varepsilon(x, y, p) \Phi_2 = 0 \quad (11)$$

Where  $\varepsilon(x, y, p) = 1 - \omega_{p_n}^2(x, y)/p^2$ . Equation (11) describes the “spatial” part of the problem. If the profile of the plasma density and the boundary conditions are specified, then solution of equation (11) gives us the desired value of the “separation constant”  $p$ . The characteristic feature of equation (11) is that the HF field amplitude does not enter it. Accordingly, equation (11) coincides with the equation describing the propagation of natural (free of external influence) electrostatic surface waves in a 2D nonuniform plane plasma waveguide.

The second set of equations (5) and (6) containing time-dependent functions  $\Phi_1$  and  $v_{\alpha_1}$  (“temporal” part of the problem) can be transformed to the desired form by eliminating  $\Phi_1$  by the use of equation (8),

$$\Phi_1(t) = \frac{4\pi}{\alpha_s} \left( v_{e_1} \frac{\alpha_e m_i}{\alpha_i m_e} e^{iA_e} + v_{i_1} e^{iA_i} \right) \quad (12)$$

and  $\alpha_s$  is given by relation (10). As a result the set of second parts of equations (5)

and (6) takes the form

$$\left. \begin{aligned} \frac{d^2 w_{e_1}}{dt^2} - i \Delta_y \frac{dw_{e_1}}{dt} + \eta_y w_{e_1} + p_e^2 (w_{e_1} + v_{i_1} e^{-iasin(\omega_0 t)}) &= 0 \\ \frac{d^2 v_{i_1}}{dt^2} + p_i^2 (v_{i_1} + w_{e_1} e^{-iasin(\omega_0 t)}) &= 0 \end{aligned} \right\} \quad (13)$$

Where;  $p_e \equiv p$ ,  $p_i \equiv (m_e/m_i) p$  and  $w_{e_1} = v_{e_1} \frac{\alpha_e m_i}{\alpha_i m_e}$ .

Comparing the derived system of equations with those describing volumetric oscillations in uniform plasma [8, 9], we find that the plasma nonuniformity results in renormalization of the plasma frequencies:  $\omega_{p_s}^2 \rightarrow p_s^2$ ,  $\omega_{p_i}^2 \rightarrow p_i^2$ . This fact enables us to use the method developed in [9] for solving the system of equations with periodical coefficients (equations (13)). At  $\gamma = 1$ , and  $V_{th} = 0$  equations (13) are in agreement with the corresponding equations for nonrelativistic cold plasma case [6].

#### 4. Solution of “temporal” equations

The solution of equations (13), according to general theory (see e.g., [20]), can be represented as follows:

$$v_{\alpha_i} = e^{-i\omega_g t} \sum_{n=-\infty}^{\infty} u_{\alpha}^{(n)} e^{-in\omega_0 t} \quad (14)$$

Using the Jacobi-Anger formula [21]:

$$e^{\pm i a \sin(\omega_0 t)} = \sum_{m=-\infty}^{\infty} J_m(a) e^{\pm i m \omega_0 t}, \quad (15)$$

We obtain the following coupled equations for the quantities  $u_{s,i}^{(n)}$  [8, 9]:

$$\left. \begin{aligned} u_s^{(n)} + R_s^{(n)}(p) \sum_{m=-\infty}^{\infty} u_i^{(m)} J_{n-m}(a) &= 0 \\ u_i^{(n)} + R_i^{(n)}(p) \sum_{m=-\infty}^{\infty} u_s^{(m)} J_{n-m}(a) &= 0 \end{aligned} \right\} \quad (16)$$

Where;

$$\left. \begin{aligned} R_{\alpha}^{(n)}(p) &= \frac{\delta \varepsilon_{\alpha}(\omega_g + n\omega_0, p)}{1 + \delta \varepsilon_{\alpha}(\omega_g + n\omega_0, p)} \\ \delta \varepsilon_s(\omega_g + n\omega_0, p) &= -\frac{p_s^2}{(\omega_g + n\omega_0)^2 + \Delta_{\gamma}^2} \\ \delta \varepsilon_i(\omega_g + n\omega_0, p) &= -\frac{p_i^2}{(\omega_g + n\omega_0)^2} \end{aligned} \right\} \quad (17)$$

Eliminating the functions  $u_i^{(n)}$ , we get

$$u_s^{(n)} = R_s^{(n)}(p) \sum_{m=-\infty}^{\infty} J_{n-m}(a) R_i^{(m)}(p) \sum_{l=-\infty}^{\infty} J_{l-n}(a) u_s^{(l)} \quad (18)$$

The infinite determinant of this system of equations, set equal to zero, constitutes the desired equation connecting the quantities  $p$  and  $\omega_g$ . For future analysis it is convenient to introduce the resonant frequencies of electrons and ions ( $\omega_{r_s}$  and  $\omega_{r_i}$ ), defined by the equations:

$$1 + \delta \varepsilon_{\alpha}(\omega_{g r_{\alpha}}, p) = 0 \quad (19)$$

We confine further analysis to the case  $\omega_0 \gg \omega_{r_i}$  ( $R_i^{(m)} \ll 1$ ). Below we shall distinguish between the resonant ( $|(n\omega_0)^2 - \omega_{r_s}^2| \ll \omega_{r_s}^2$ ) and nonresonant ( $n\omega_0 \neq \omega_{r_s}$ ) cases and between the low-frequency branch of oscillations ( $\omega_{LF} \sim \omega_{r_i}$ ) and the high-frequency branch ( $\omega_{HF} \sim \omega_{r_s}$ ).

**For high-frequency oscillations** it follows from (17) that  $R_i^{(n)} \ll 1$

$$R_i^{(n)} \sim \delta \varepsilon_i(p), \delta \varepsilon_i(n \neq 0) \sim \frac{p_i^2}{(n\omega_n)^2} \sim \frac{m_s}{m_i} \ll 1, \delta \varepsilon_i(n = 0) \sim \frac{p_i^2}{\omega_{HF}^2} \sim \frac{p_i^2}{p_s^2} \ll 1 \quad (20)$$

Therefore, solution of equation (18) exists only at  $R_s^{(n)} \gg 1$  for certain  $n$ . It follows from equation (18) that the largest  $u_s^{(l)}$  from the second sum in (18) corresponds to  $l = n$ . In such case equation (18) yields:

$$R_s^{(n)}(p) \sum_{m=-\infty}^{\infty} J_{n-m}^2(a) R_i^{(m)}(p) = 1 \quad (21)$$

**For low-frequency oscillations**  $R_s^{(m)} \sim 1$  for all  $m$  ( $n\omega_0 \neq \omega_{gr_s}$ ). Among the coefficients  $R_i^{(m)}$  the greatest is  $R_i^{(0)} \gg R_i^{(m \neq 0)}$ . Therefore, we can replace equation (18) by:

$$u_s^{(n)} = R_s^{(n)}(p) R_i^{(0)}(p) J_n(a) \sum_{l=-\infty}^{\infty} J_l(a) u_s^{(l)} \quad (22)$$

From equation (22) we get

$$R_i^{(0)}(p) \sum_{n=-\infty}^{\infty} J_n^2(a) R_s^{(n)}(p) = 1 \quad (23)$$

As a consequence of the dependence of values  $R_s^{(n)}$  on the separation constant  $p$ , equations (21) and (23) describe the influence of a HF electric field on the propagation characteristics of electrostatic waves in a nonuniform plasma waveguide. In case of uniform plasma, equations (21) and (23) have been derived for the first time in [8].

Let us analyze equations (22) and (23) for different values of the pump field frequency  $\omega_0$ .

#### 4. 1. Ultra-high-frequency pump field ( $\omega_0 \gg p_s$ )

Since  $\delta \varepsilon_s \sim \frac{p_s^2}{(n\omega_n)^2} \ll 1$ , from (17) we get  $R_s^{(0)} \gg R_s^{(n \neq 0)}$ . In this case equation (21) assumes the form:

$$(\omega_s^2 - p_s^2 - \Delta_v^2 + \eta_v^2)(\omega_s^2 - p_i^2) = p_s^2 p_i^2 J_0^2(a) \quad (24)$$

Taking into account that the ratio  $(p_i^2/p_e^2) \sim (m_e/m_i)$ , we find from (24) two relations connecting values of constant  $p$  with frequency of externally generated electrostatic wave  $\omega_a$ .

**a) High-frequency range ( $\omega_a^{HF} \approx p_e$ )**

$$\omega_a^{HF} \approx p_e^{HF} \left( \beta_\gamma^2 + \frac{m_e}{m_i} J_n^2(a) \right)^{1/2} \quad (25)$$

$$\text{Where; } \beta_\gamma^2 = 1 + \frac{(\Delta_\gamma^2 - \eta_\gamma^2)}{p_e^2}.$$

In this range the frequency  $\omega_a^{HF}$  only slightly differs from the value  $p_e^{HF}$ .

At  $\gamma = 1$ , and  $V_{th} = 0$ , equation (25) is in agreement with the corresponding equations for nonrelativistic cold plasma case [6].

**b) Low-frequency range ( $\omega_a^{LF} \approx p_i$ )**

$$\omega_a^{LF} \approx p_i^{LF} (1 - J_0^2(a))^{1/2} \quad (26)$$

The value  $\omega_a^{LF}$  results from the action of HF electric field and disappears as amplitude  $E_0$  vanishes.

**4. 2. Parametric resonance ( $n\omega_0 \approx p_e$ )**

**a) High-frequency range ( $\omega_a^{HF} \approx p_e$ )**

We have from (17)

$$R_e^{(0)} = - \frac{p_e^2}{(\omega_a^{HF})^2 - (p_e^2 + \Delta_\gamma^2 - \eta_\gamma^2)} \gg R_e^{(n \neq 0)} \sim \frac{p_e^2}{(n\omega_0)^2 + \Delta_\gamma^2 - \eta_\gamma^2},$$

$$R_i^{(m)} \approx - \frac{p_i^2}{(m\omega_0 + \omega_a^{HF})^2}$$

Equation (21) reduces to

$$\omega_a^{HF} \approx p_e^{HF} \left( \beta_\gamma^2 + \frac{1}{2} \sum_{m=-\infty}^{\infty} J_m^2(a) \frac{p_i^2}{(p_e + m\omega_0)^2} \right) \quad (27)$$

From (27) it follows that accounting of an HF electric field and also due to the relativistic warm electrons lead to small corrections to the frequency of externally generated plasma wave. In conclusion, relativistic of the electrons plasma lead to a



decrease of the value of the reflection coefficient in comparison with the case of nonrelativistic cold plasma ( $\gamma = 1$  and  $V_{th} = 0$ ) [6].

**b) Low-frequency range ( $\omega_g^{LF} \approx p_i$ )**

For this range of frequency from (17) we find

$$\delta \varepsilon_i (n \neq 0) \sim \frac{p_i^2}{(n\omega_0)^2} \ll 1, \quad R_i^{(0)} \approx -\frac{p_i^2}{\omega_g^{LF2}} \gg R_i^{(n \neq 0)},$$

$$R_\varepsilon^{(n)} = -\frac{p_\varepsilon^2}{(n\omega_0 + \omega_g^{LF})^2 - (p_\varepsilon^2 - \Delta_\gamma^2 + \eta_\gamma^2)}$$

Thus equation (23) becomes

$$(\omega_g^{LF})^2 = (p_i^{LF})^2 \sum_{n=-\infty}^{\infty} J_n^2(a) \frac{(p_\varepsilon^{LF})^2}{(n\omega_0 + \omega_g^{LF})^2 - (p_\varepsilon^{LF})^2} \quad (28)$$

Under the condition  $\left| (n\omega_0 + \omega_g^{LF})^2 - (p_\varepsilon^2 - \Delta_\gamma^2 + \eta_\gamma^2) \right| \ll p_\varepsilon^2$  it is necessary to keep only two resonant terms from the sum over  $n$  in equation (28). As a result we get

$$(p_{1,2}^{LF})^2 = \frac{1}{\sigma_{1n}} \{ \sigma_{2n} \pm [\sigma_{2n}^2 - \sigma_{1n} C]^{1/2} \} \quad (29)$$

Where;

$$\sigma_{1n} = (\omega_g^{LF})^2 + 2 \frac{m_\varepsilon}{m_i} J_n^2(a) \left( (\omega_g^{LF})^2 + (n\omega_0)^2 \right),$$

$$\sigma_{2n} = (\omega_g^{LF})^2 \left( (\omega_g^{LF})^2 + (n\omega_0)^2 - \Delta_\gamma^2 + \eta_\gamma^2 \right)$$

and

$$C = (\omega_g^{LF})^2 \left( \left( (\omega_g^{LF})^2 - (n\omega_0)^2 \right)^2 - \Delta_\gamma^4 + \eta_\gamma^4 + 2 (\Delta_\gamma^2 + \eta_\gamma^2) \left( (\omega_g^{LF})^2 + (n\omega_0)^2 \right) \right)$$

At  $\gamma = 1$  and  $V_{th} = 0$ , equation (29) is in agreement with the corresponding equations for nonrelativistic cold plasma case [6].

## 5. Solution of “spatial” equations

Taking into account expression (1) it is possible to perform separation variables in equation (10) ( $\Phi_2(x, y) = f_1(x)f_2(y)$ ). For the functions  $f_1(x)$  and  $f_2(y)$  we obtain:

$$\frac{d^2 f_1}{dx^2} + \left( k^2 + \frac{\chi_0^2}{1 + e^{x/L}} \right) f_1 = 0 \quad (30)$$

$$\frac{d^2 f_2}{dy^2} + k_y^2 f_2 = 0, \quad (31)$$

Where;  $k^2 = k_{\parallel}^2 |\varepsilon_0(p)| - k_y^2 > 0$ ,  $\chi_0^2 = \frac{k_{\parallel}^2 \omega_{p_0}^2 \mu}{p^2}$ ,  $\omega_{p_0}^2 = \frac{4\pi e^2 N}{m}$ , and  $\varepsilon_0(p) = 1 - \frac{\omega_{p_0}^2}{p^2}$ .

### 5. 1. Traveling wave solution

Let us consider the primarily solution of equation (30) corresponding to a traveling wave. Following [6], we obtain the reflection coefficient:

$$R^2 = \frac{|c_2|^2}{|c_1|^2} = \frac{\sinh^2[\pi(\chi - k)L]}{\sinh^2[\pi(\chi + k)L]} \quad (32)$$

Where;  $c_1 = \frac{\Gamma(1-2ikL)\Gamma(-2i\chi L)}{\Gamma(-i(\chi+k)L)\Gamma(1-i(\chi+k)L)}$ ,  $c_2 = \frac{\Gamma(1-2ikL)\Gamma(2i\chi L)}{\Gamma(i(\chi+k)L)\Gamma(1+i(\chi+k)L)}$ . ( $\Gamma$  is Euler's gamma function)

We note that in the limiting case of uniform plasma ( $\mu = 0$ ), the reflection coefficient  $R = 0$  ( $\chi = k$ ), i.e., in this case the wave does not suffer a reflection.

The separation constant  $p$  enters into expressions (32) through the quantities  $k$  and  $\chi$ . It is evident from expressions (25) – (26) that the influence of a HF electric field results in modifications of the propagation constants  $k$  and  $\chi$ . This change in turn will cause modifications in the value of the reflection coefficient.

By way of illustrations we shall treat expression (32) assuming presence of ultra-high-frequency pumps field ( $\omega_0 \gg p_e$ ). In the limiting case ( $\chi - k \ll 1$ ) expression (32) reduces to

$$R = \pi \mu (k_{\parallel} L)^2 \frac{\omega_{p_0}^2}{p^2} \frac{1}{2k_0 L} \sinh^{-1}(2\pi k_0 L) \quad (33)$$

Where;  $k_0 = k$  ( $\mu = a = 0$ ).

#### a) High-frequency range ( $\omega_q^{HF} \approx p_e$ )

Using expression (25), under condition  $a \ll 1$  we get from (33)

$$R \approx \pi \frac{\omega_{p_0}^2}{(\omega_q^{HF})^2} \frac{(k_{\parallel} L)^2}{2k_0 L} \frac{\mu_{eff}}{\sinh(2\pi k_0 L)} \quad (34)$$

$$\mu_{eff} \approx \beta_\gamma^2 + \frac{\omega_{pe}^2}{\omega_{pe}^2} \left( 1 - \frac{k_{\parallel}^2 r_E^2}{4} \right), \quad r_E = \frac{\varepsilon E_0}{m_e \omega_0^2} \quad (35)$$

From expressions (34) and (35) it follows that a HF electric field and the relativistic electrons plasma reduce the value of the reflection coefficient.

At  $\gamma = 1$ , and  $V_{th} = 0$  equation (35) is in agreement with the corresponding equations for nonrelativistic cold plasma case [6].

### b) Low-frequency range ( $\omega_g^{LF} \approx p_i$ )

In this case from (33) and (26) it follows

$$R \approx \pi \mu (k_{\parallel} L)^2 \frac{\omega_{pe}^2}{(\omega_g^{LF})^2} (1 - J_0^2(a)) \frac{\sinh^{-1}(2\pi k_0 L)}{2k_0 L} \quad (36)$$

At  $a \ll 1$  expression (36) reduces to:

$$R \approx \pi \mu (k_{\parallel} L)^2 \frac{\omega_{pe}^2}{4(\omega_g^{LF})^2} (k_{\parallel}^2 r_E^2) \frac{\sinh^{-1}(2\pi k_0 L)}{2k_0 L} \quad (37)$$

In a given frequency range, as one would expect, reflection coefficient tends to zero at vanishing HF field amplitude since a wave the frequency  $\omega_g^{LF}$  can propagate in a plasma waveguide only at  $E_0 \neq 0$  (see expression (26)).

## 5. 2. Electric field potential distribution

From equation (31) we find consequently [22]:

$$f_2^s(y) = f_{20}^s \cos(k_y^s y), \quad k_y^s d = \frac{\pi}{2} (2m + 1), \quad (38)$$

$$f_2^{as}(y) = f_{20}^{as} \sin(k_y^{as} y), \quad k_y^{as} d = m\pi, \quad (39)$$

Where;  $m = 0, 1, 2, \dots$ . Superscripts *s* and *as* denote symmetric and axisymmetric respectively. Expressions (38) and (39) determine both the spatial distribution of the electric potential in a cross-section of a plasma waveguide and the values of the transversal wave number  $k_y$ .

## 6. Results and conclusions

Investigating in this paper the problem of propagation of an electrostatic wave in a bounded nonuniform relativistic warm plasma waveguide. Using a new

mathematical technique “separation method” for the solution of an eigenvalue problem when the real wave number  $k$  is a function of the real frequency. In fact we treat a problem of forced oscillation excited in plasma by external source (generator) with a fixed frequency.

It is found (expressions (25), (26), (27) and (28)) the influence of the HF electric field and the relativistic warm electrons plasma result in modifications of the propagation constants which in turn causes a modification in the value of the reflection coefficient.

It is proved that when the frequency of incident wave  $\omega_g$  close to the relativistic warm electron plasma frequency  $\omega_{pe}$ , plasma is less reflective due to presences of HF field and the relativistic warm electrons. On the other hand, for low-frequency incident wave ( $\omega_g \sim \omega_{pe}$ ) the reflection coefficient is directly proportional to the amplitude of the HF field.

It is also found that (expressions (34), and (35)), relativistic electrons plasma lead to a decrease of the value of the reflection coefficient in comparison with the case of nonrelativistic cold plasma (i.e., at  $\gamma = 1$  and  $V_{th} = 0$ ) [6] and relativistic cold plasma [23]. The proposed approach (“separation method”) is significantly simpler than the method ordinarily employed in the theory of parametric resonance in a nonuniform plasma (e.g., ref. [10] and references therein). Therefore it is of special interest to apply the separation method to solve different problems involving parametric excitation of electrostatic waves in bounded nonuniform plasma.

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**Received: May, 2009**