**Adv. Studies Theor. Phys., Vol. 4, 2010, no. 1, 47 - 54**

# $n$-Fold Kronecker Products of $\mathfrak{su}(2)$-Multiplets and $\mathfrak{h}_N$-Compatibility

**R. Campoamor-Stursberg**

I.M.I. and Dpto. Geometría y Topología
Universidad Complutense de Madrid
Plaza de Ciencias 3, E-28040 Madrid
rutwig@pdi.ucm.es

**Abstract**

We classify the Kronecker products $D^{j_1} \otimes ... \otimes D^{j_n}$ of $\mathfrak{su}(2)$-multiplets that are compatible with Heisenberg-Weyl algebras by means of the analysis of antisymmetric Kronecker products. Alternative formulae for the decomposition of the latter are obtained.

**PACS:** 02.20Sv

**Keywords:** antisymmetric Kronecker product, Lie algebra, Heisenberg-Weyl algebra

## 1 Introduction

Kronecker products of irreducible representations of Lie algebras play a key role in any labelling problem, as well as in the study of the corresponding branching rules. The problem of decomposing square Kronecker products of irreducible multiplets of Lie algebras has also become a central tool to explain unsuspected symmetries behind the vanishing of matrix elements, also known as the problem of conflicting symmetries [1]. Motivated by these problems, a systematic program to evaluate Kronecker products and resolve symmetrized powers for all physically relevant groups was initiated ([1] and references therein). Besides its wide interest in the labelling problem, symmetric Kronecker products are also important for the classification of various properties of representations, as well as for the study of semidirect products of Lie algebras. In particular, the structure of semidirect products of semisimple and Heisenberg-Weyl algebras can be determined by using this method.

In this note we extend the analysis of the compatibility problem, as developed in [2], to the case of arbitrary Kronecker products of irreducible $\mathfrak{su}(2)$-

Table 1: Symmetric Kronecker products of $\mathfrak{su}(2)$-multiplets

| | $\bigwedge^2 D^j$ | $\dim \bigwedge^2 D^j$ | $\mathrm{Sym}^2 D^j$ | $\dim \mathrm{Sym}^2 D^j$ |
|---|---|---|---|---|
| $j=\frac{1}{2},\frac{3}{2},..$ | $\sum_{\alpha=0}^{2j-\frac{1}{2}} D^{(2j-2\alpha)}$ | $j\,(2j+1)$ | $\sum_{\alpha=0}^{2j-\frac{1}{2}} D^{(2j-2\alpha-1)}$ | $(j+1)\,(2j+1)$ |
| $j=0,1,2,..$ | $\sum_{\alpha=0}^{j-2} D^{(2j-2\alpha)}$ | $j\,(2j+1)$ | $\sum_{\alpha=0}^{j} D^{(2j-2\alpha)}$ | $(j+1)\,(2j+1)$ |

multiplets. This will lead to alternative decomposition formulae for the antisymmetric Kronecker products of such representations, as well as for their decomposition into irreducible multiplets. The corresponding structure constants for the corresponding semidirect products are also found.

To this extent, we will make use of the standard realization of $\mathfrak{su}(2)$ in terms of creation and annihilation operators $a_i, a_j^*$ satisfying the commutators

$$[a_i, a_j] = \left[a_i^*, a_j^*\right] = 0,\ \left[a_i, a_j^*\right] = \delta_{i,j}.$$

The boson realization is given by the operators $J_0 = \frac{1}{2}\left(a_1^* a_1 - a_2^* a_2\right),\ J_\pm = \frac{1}{2}\left((1\pm 1)\,a_1^* a_2 + (1\mp 1)\,a_2^* a_1\right)$ with commutators

$$[J_+, J_-] = 2J_0,\ [J_0, J_\pm] = \pm J_\pm$$

The $(2j+1)$-dimensional (irreducible) multiplets $D^j$ of $\mathfrak{su}\,(2)$ are given by $\{\,|j,m\rangle \mid m=-j,-j+1,..,j-1,j\}$, where the state $|j,m\rangle$ is specified by

$$|j,m\rangle = \frac{\left(a_1^*\right)^{j+m}\left(a_2^*\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\,|0\rangle\,.$$

Since the eigenvalues of $J_0$ are always symmetric with respect to zero, it follows at once that any $D^j$ is self-dual, i.e., isomorphic to its contragredient representation. As known, for self-dual representations the square Kronecker product always contains a copy of the identity representation [3]. In particular, for the Kronecker product of a multiplet with itself we get the decomposition

$$D^j \otimes D^j = \sum_{k=0}^{2j} D^{(2j-\alpha)} = \mathrm{Sym}^2 D^j \oplus \bigwedge^2 D^j. \tag{1}$$

The corresponding division into symmetric and antisymmetric parts is given in Table 1.

## 1.1 Compatibility of multiplets

We convene that for arbitrary multiplets $\Gamma, \Gamma'$ of $\mathfrak{su}\,(2)$, the symbol $\mathrm{mult}_\Gamma(\Gamma')$ denotes the multiplicity of $\Gamma'$ in $\Gamma$, i.e., the number of copies of $\Gamma$ appearing the the decomposition $\Gamma = \bigoplus D^j$ into irreducible multiplets. We call the $D^j$ the constituents of $\Gamma$. Following [2], we say that a $2N$-dimensional representation $\Gamma$ of $\mathfrak{su}\,(2)$ is compatible with the $(2N+1)$-dimensional Heisenberg-Weyl algebra $\mathfrak{h}_N$ if the constituents $D^j$ of $\Gamma$ satisfy one of the following conditions[1]

**C1.** if $D^j \wedge D^j \not\supseteq D^0$, then $\mathrm{mult}_\Gamma\,(D^j)$ is even,

**C2.** if $\mathrm{mult}_\Gamma\,(D^j)$ is odd, then $D^j \wedge D^j \supseteq D^0$.

The compatibility of multiplets in particular implies the existence [2] of a semidirect product Lie algebra $\mathfrak{g}$ with Levi decomposition

$$\mathfrak{g} = \mathfrak{su}\,(2)\ \overrightarrow{\oplus}_{\Gamma\oplus D^0}\mathfrak{h}_N, \tag{2}$$

where $D^0 = |0,0\rangle$ is the identity representation. The latter class of algebras is of wide interest in the context of stability theory of Lie algebras,[2] as well as in the study of boson realizations of Lie algebras [5, 6].

In particular, if we restrict to irreducible multiplets $D^j$, it follows at once from Table 1 that only those $D^j$ with half-integer $j$ are compatible with a Heisenberg-Weyl algebra. For multiplets with integer $j$, condition **C1** should be used [2]. For direct sums of these multiplets, analogous results hold[2]. It is therefore natural to ask whether the situation can be generalized to Kronecker products $\Gamma = D^{j_1}\otimes ...\otimes D^{j_n}$ of irreducible multiplets, i.e., under which circumstance they are compatible with some Heisenberg-Weyl algebra satisfying the condition **C2**. The compatibility problem therefore reduces to the study of the antisymmetric Kronecker product of $\Gamma$ and its decomposition into irreducible multiplets.

Let $n=2$ and suppose that $j \geq k$. The Kronecker product decomposes as $D^j \otimes D^k = \sum_{\alpha=0}^{k} D^{(j+k-2\alpha)}$. Using (1), it is easy to derive the decomposition $\bigwedge^2\left(D^j\otimes D^k\right) = \left(\mathrm{Sym}^2 D^j \otimes \bigwedge^2 D^k\right) \oplus \left(\bigwedge^2 D^j \otimes \mathrm{Sym}^2 D^k\right)$. Now observe that in order to satisfy the condition **C2**, one of the indices must be a half-integer, while the other must be integer. In particular, this will imply that in the decomposition of $D^j\otimes D^k$ into irreducible multiplets, all indices will be half-integers.

For $n=3$, the antisymmetric Kronecker product $\bigwedge^2\left(D^{j_1}\otimes D^{j_2}\otimes D^{j_3}\right)$ has dimension

$$d = \frac{\left(\left(2j_1+1\right)\left(2j_2+1\right)\left(2j_3+1\right)\right)\left(\left(2j_1+1\right)\left(2j_2+1\right)\left(2j_3+1\right)-1\right)}{2}.$$

Using the elementary properties of Kronecker products [3] and equation (1), we rewrite the product of $\left(D^{j_1}\otimes D^{j_2}\otimes D^{j_3}\right)$ with itself as the direct sum

[1] In the general case there is a third condition. Since for $\mathfrak{su}\,(2)$ all irreducible multiplets are self-dual, we merely get two constraints.

[2] The probably best known example of this type is the Carroll Lie algebra in dimension 10 [4].

$\bigoplus_{l=1}^{3}\left(\mathrm{Sym}^2 D^{j_l}\otimes\bigwedge^2 D^{j_l}\right)$. Expanding the latter product, it follows that the terms with an odd number of wedge products are those corresponding to the antisymmetric Kronecker product. We therefore get

$$\bigwedge^2\left(D^{j_1}\otimes D^{j_2}\otimes D^{j_3}\right)=\frac{1}{2}\sum_{\sigma\in S_3}\mathrm{Sym}^2 D^{j_{\sigma(1)}}\otimes\mathrm{Sym}^2 D^{j_{\sigma(2)}}\otimes\bigwedge^2 D^{j_{\sigma(3)}}$$
$$\oplus\frac{1}{6}\sum_{\sigma\in S_3}\bigwedge^2 D^{j_{\sigma(1)}}\otimes\bigwedge^2 D^{j_{\sigma(2)}}\otimes\bigwedge^2 D^{j_{\sigma(3)}}.$$

This shows that there are two possible configurations of the indices $\{j_1, j_2, j_3\}$ such that the Kronecker product is compatible:

1. the indices $j_1, j_2$ are integers and $j_3$ is a half-integer (or some permutation of $\{1,2,3\}$)

2. the indices $j_1, j_2, j_3$ are all half-integers.

# 2 Kronecker products $D^{j_1}\otimes ...\otimes D^{j_n}$

In analogy with the cases $n=2$ and $n=3$, we can obtain the decomposition of the arbitrary Kronecker product $D^{j_1}\otimes ...\otimes D^{j_n}$ developing the Kronecker product with the help of decomposition (1), and then expanding the sum. It is clear that the process is iterative, thus the proof will follow by recurrence. An alternative way to prove the assertion can be obtained using the Young diagrams [3].

**Lemma 1** *For $n\geq 2$, the antisymmetric Kronecker product of $D^{j_1}\otimes ...\otimes D^{j_n}$ decomposes as*

$$\bigwedge^2\left(D^{j_1}\otimes ...\otimes D^{j_n}\right)=\sum_{\sigma\in S_n}\sum_{k=0}^{\left[\frac{n+1}{2}\right]}\frac{1}{(2k+1))!\ (n-2k-1)!}\times$$
$$\mathrm{Sym}^2 D^{j_{\sigma(1)}}\otimes ..\otimes\mathrm{Sym}^2 D^{j_{\sigma(n-2k-1)}}\otimes\bigwedge^2 D^{\sigma(n-2k)}\otimes ..\otimes\bigwedge^2 D^{\sigma(n)},\qquad(3)$$

*where $S_n$ denotes the symmetric group in $n$ letters.*

With the use of formula (3), we can determine easily which configurations of the indices $\{j_1, .., j_n\}$ will lead to Kronecker products $D^{j_1}\otimes ...\otimes D^{j_n}$ that satisfy the compatibility condition **C2**.

**Lemma 2** *Let $k = 0, 1, .., \left[\frac{n+1}{2}\right]$. Then $\bigwedge^2 (D^{j_1} \otimes ... \otimes D^{j_n})$ contains a copy of $D^0$ if there exist half-integers $j_{l_1}, ..., j_{l_{2k+1}} \in \{j_1, .., j_n\}$, and any other index $j_p \notin \left\{j_{l_1}, ..., j_{l_{2k+1}}\right\}$ is an integer. In particular, for $k = \left[\frac{n-1}{2}\right]$, we distinguish two cases according to the parity of $n$:*

1. *If $n$ is even, then all $j_i'$s but one are half-integers.*
2. *If $n$ is odd, then all the indices $j_i$ are half-integers.*

The proof follows at once using the properties of $\mathfrak{su}(2)$-multiplets. If $j_l$ is a half-integer, then we have $\bigwedge^2 D^{j_l} \supset D^0$, while for integer indices $j_l$ we get $\mathrm{Sym}^2 D^{j_l} \supset D^0$. Therefore, for a fixed $k = 0, .., \left[\frac{n+1}{2}\right]$, the sum

$$\sum_{\sigma\in S_n} \mathrm{Sym}^2 D^{j_{\sigma(1)}} \otimes .. \otimes \mathrm{Sym}^2 D^{j_{\sigma(n-2k-1)}} \otimes \bigwedge^2 D^{\sigma(n-2k)} \otimes .. \otimes \bigwedge^2 D^{\sigma(n)}$$

contains the identity representation only if we can find indices of $\{j_1, .., j_n\}$ such that

$$\mathrm{Sym}^2 D^{j_l'} \supset D^0, \ \bigwedge^2 D^{j_m'} \supset D^0$$

where $l = 1, .., n-2k-1$ and $m = n-2k, .., n$. This implies that $\{j_1, .., j_n\}$ admits a partition into $\left\{j_1', .., j_{n-2k-1}'\right\}$ integers and $\left\{j_{n-2k}', .., j_n'\right\}$ half-integers, and we can therefore find a permutation $\sigma \in S_n$ such that $j_l' = j_{\sigma(l)}, \ j_m' = j_{\sigma(m)}$.

Since the scalar $k$ can take $\left[\frac{n+1}{2}\right]$ different values, for any $n \geq 2$ there will alsways be $\left[\frac{n+1}{2}\right]$ different configurations of the indices which lead to compatible Kronecker products.

Once having obtained which Kronecker products satisfy condition **C2**, it remains to obtain the decomposition into irreducible representations. As before, a long but straightforward computation, using a recursion argument, allows to obtain the decomposition of $D^{j_1} \otimes ... \otimes D^{j_n}$. It is given by the expression:[3]

$$D^{j_1} \otimes ... \otimes D^{j_n} = \sum_{\alpha_2=0}^{2j_2} \sum_{\alpha_3=0}^{\xi_1} ... \sum_{\alpha_n=0}^{\xi_{n-2}} D^{(j_1-\alpha_2+...+j_n-..-a_n)}, \tag{4}$$

where

$$\xi_i = \min\left\{ 2\left(\sum_{\beta=1}^{i-1} j_\beta - \sum_{\varepsilon=2}^{i-1} \alpha_\varepsilon\right), 2j_i \right\}. \tag{5}$$

[3]Actually this decomposition formula holds for the general case, i.e., for arbitrary indices $\{j_1, .., j_n\}$.

In particular, it follows from (4) that the constituents are always multiplets the indices of which are half-integers. This is a direct consequence of Lemma 2. In addition, the multiplicity of any multiplet $D^{k_0}$ occurring in the preceding decomposition is further obtained as the number of solutions of the linear system

$$k_0 = \sum_{l=1}^{n} j_l - \alpha_2 - .. - \alpha_n, \tag{6}$$

where $\alpha_2 = 0, .., .j_2$ and $\alpha_i = \xi_i$ for $i = 3, .., n-1$.

In Table 2, all compatible Kronecker products are given up to dimension $d = 100$. In general, for any even number $m \geq 2$ there exists an integer $n$ and irreducible multiplets $D^{j_l}$ of $\mathfrak{su}(2)$ such that $D^{j_1} \otimes ... \otimes D^{j_n}$ is compatible and has dimension $m$.

# 3 Semidirect products of Lie algebras

As we commented before, the classification of compatible representations $\Gamma$ allows to construct Lie algebras with the Levi decomposition $\mathfrak{su}(2) \overrightarrow{\oplus}_{\Gamma \oplus D^0} \mathfrak{h}_N$, where $N = \dim \Gamma$. Here we briefly indicate one of the possibilities to do it. Consider a compatible representation $\Gamma = D^{j_1} \otimes ... \otimes D^{j_n}$ and its decomposition (4). For any constituent $D^{j_1+..+j_n-\alpha}$ we denote its multiplicity by $k_0$. As basis of $\Gamma$ can be chosen as

$$X^m_{\alpha,\mu} := |j_1 + .. + j_n - \alpha, m\rangle \,, \ \mu = 1, .., k_0.$$

The index $\mu$ indicates in which copy of $D^{j_1+..+j_n-\alpha}$ the vector is placed. Since the centre generator $M$ of $\mathfrak{h}_N$ commutes with $\mathfrak{su}(2)$, the commutation relations among the vectors $X^m_{\alpha,\mu}$ must have the following form:

$$\left[X^m_{\alpha,\mu}, X^{m'}_{\alpha',\mu'}\right] = \delta^{\alpha'}_{\alpha} \delta^{m'}_{-m} M. \tag{7}$$

Observe that no restriction is made on vector belonging to different copies of the same multiplet. A special solution to (7) is therefore given if we modify the commutator by

$$\left[X^m_{\alpha,\mu}, X^{m'}_{\alpha',\mu'}\right] = \delta^{\alpha'}_{\alpha} \delta^{m'}_{-m} \delta^{\mu'}_{\mu} M.$$

In this case, only the generators of each copy of the multiplets in the decomposition (4) have a non-zero bracket. Evaluating either the Jacobi conditions or the Maurer-Cartan forms, the structure constants can be found to be

$$\left[X^m_{\alpha,\mu}, X^{-m}_{\alpha,\mu}\right] = \frac{2\,(j_1 + .. + j_n - \alpha)\,\Gamma\,(2j_1 + .. + 2j_n - 2\alpha)\,(-1)^{j_1+..+j_n-\alpha-m}}{\Gamma\,(j_1 + .. + j_n - \alpha + 2 - m)\,\Gamma\,(j_1 + .. + j_n - \alpha - m)}\, M,$$

where $\Gamma(z)$ denotes the Gamma function. These algebras constitute a natural generalization of those analyzed in [2], although other non-isomorphic possibilities can exists in dependence of the multiplicities of the constituents of (4).

# Concluding remarks

We have obtained the natural generalization of the compatibility problem with Heisenberg-Weyl algebras for arbitrary Kronecker products $D^{j_1} \otimes ... \otimes D^{j_n}$ of $\mathfrak{su}(2)$ representations. In particular, the general formula for the antisymmetric tensor products and its decomposition into irreducible components have been derived. It has also been pointed out that any compatible multiplet gives rise to a semidirect product of Lie algebras. Among the various possibilities, that generalizing naturally those of [2] have been determined.

These results have potential application in the corresponding compatibility problem for higher rank simple Lie algebras, as well as the systematic study of the symmetric Kronecker products of their (irreducible) representations. This case contains quite interesting cases, like the Schrödinger Lie algebras [7]. Another possible application is given by the branching rules for the labelling problems with respect to $\mathfrak{su}(2)$ subalgebras [8].

**ACKNOWLEDGEMENTS.** This work was partially supported research project GR58/4120818-920920 of the UCM-BSCH.

# References

[1] B. G. Wybourne. *Lit. J. Phys.* **35** (1995), 123

[2] R. Campoamor-Stursberg. *Acta Phys.* **B36** (2005), 2869

[3] B. G. Wybourne, *Classical Groups for Physicists*, Wiley Interscience, New York 1974.

[4] J. F. Cariñena, M. A. del Olmo and M. Santander. *J. Phys. A: Math. Gen.* **14** (1981), 1

[5] A. Klein and E. R. Marshalek. *Rev. Mod. Phys.* **63** (1991), 375

[6] R. Campoamor-Stursberg. *J. Phys. A: Math. Gen* **38** (2005), 4187

[7] P. Feinsilver, J. Kocik and R. Schott, *Fort. Physik* **52** (2004), 343

[8] P. D. Jarvis and R. B. Zhang. *J. Phys. A: Math. Gen.* **38**, L219

**Received: August, 2009**

Table 2: Compatible $\mathfrak{su}(2)$-multiplets $R$ of dimension $d \leq 100$

| $R$ | $d$ | $R$ | $d$ | $R$ | $d$ | $R$ | $d$ |
|---|---|---|---|---|---|---|---|
| $D^{\frac{1}{2}} \otimes D^{1}$ | 6 | $D^{\frac{3}{2}} \otimes D^{11}$ | 92 | $D^{\frac{29}{2}} \otimes D^{1}$ | 90 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{5}{2}}$ | 48 |
| $D^{\frac{1}{2}} \otimes D^{2}$ | 10 | $D^{\frac{3}{2}} \otimes D^{12}$ | 100 | $D^{\frac{31}{2}} \otimes D^{1}$ | 96 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{7}{2}}$ | 64 |
| $D^{\frac{1}{2}} \otimes D^{3}$ | 14 | $D^{\frac{5}{2}} \otimes D^{1}$ | 18 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{1}$ | 18 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{9}{2}}$ | 80 |
| $D^{\frac{1}{2}} \otimes D^{4}$ | 18 | $D^{\frac{5}{2}} \otimes D^{2}$ | 30 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{2}$ | 30 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{11}{2}}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{5}$ | 22 | $D^{\frac{5}{2}} \otimes D^{3}$ | 42 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{3}$ | 42 | $D^{\frac{1}{2}} \otimes D^{\frac{5}{2}} \otimes D^{\frac{5}{2}}$ | 72 |
| $D^{\frac{1}{2}} \otimes D^{6}$ | 26 | $D^{\frac{5}{2}} \otimes D^{4}$ | 54 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{4}$ | 54 | $D^{\frac{1}{2}} \otimes D^{\frac{5}{2}} \otimes D^{\frac{7}{2}}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{7}$ | 30 | $D^{\frac{5}{2}} \otimes D^{5}$ | 66 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{5}$ | 66 | $D^{\frac{3}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{3}{2}}$ | 64 |
| $D^{\frac{1}{2}} \otimes D^{8}$ | 34 | $D^{\frac{5}{2}} \otimes D^{6}$ | 78 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{6}$ | 78 | $D^{\frac{3}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{5}{2}}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{9}$ | 38 | $D^{\frac{5}{2}} \otimes D^{7}$ | 90 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{7}$ | 90 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{1} \otimes D^{1}$ | 54 |
| $D^{\frac{1}{2}} \otimes D^{10}$ | 42 | $D^{\frac{7}{2}} \otimes D^{1}$ | 24 | $D^{\frac{1}{2}} \otimes D^{2} \otimes D^{2}$ | 50 | $D^{\frac{1}{2}} \otimes D^{1} \otimes D^{1} \otimes D^{2}$ | 90 |
| $D^{\frac{1}{2}} \otimes D^{11}$ | 46 | $D^{\frac{7}{2}} \otimes D^{2}$ | 40 | $D^{\frac{1}{2}} \otimes D^{2} \otimes D^{3}$ | 70 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{1}$ | 24 |
| $D^{\frac{1}{2}} \otimes D^{12}$ | 50 | $D^{\frac{7}{2}} \otimes D^{3}$ | 56 | $D^{\frac{1}{2}} \otimes D^{2} \otimes D^{4}$ | 90 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{2}$ | 40 |
| $D^{\frac{1}{2}} \otimes D^{13}$ | 54 | $D^{\frac{7}{2}} \otimes D^{4}$ | 72 | $D^{\frac{1}{2}} \otimes D^{3} \otimes D^{3}$ | 98 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{3}$ | 56 |
| $D^{\frac{1}{2}} \otimes D^{14}$ | 58 | $D^{\frac{7}{2}} \otimes D^{5}$ | 88 | $D^{\frac{3}{2}} \otimes D^{1} \otimes D^{1}$ | 36 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{4}$ | 72 |
| $D^{\frac{1}{2}} \otimes D^{15}$ | 62 | $D^{\frac{9}{2}} \otimes D^{1}$ | 30 | $D^{\frac{3}{2}} \otimes D^{1} \otimes D^{2}$ | 60 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{5}$ | 88 |
| $D^{\frac{1}{2}} \otimes D^{16}$ | 66 | $D^{\frac{9}{2}} \otimes D^{2}$ | 50 | $D^{\frac{3}{2}} \otimes D^{1} \otimes D^{3}$ | 84 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{1}$ | 48 |
| $D^{\frac{1}{2}} \otimes D^{17}$ | 70 | $D^{\frac{9}{2}} \otimes D^{3}$ | 70 | $D^{\frac{3}{2}} \otimes D^{2} \otimes D^{2}$ | 100 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{2}$ | 80 |
| $D^{\frac{1}{2}} \otimes D^{18}$ | 74 | $D^{\frac{9}{2}} \otimes D^{4}$ | 90 | $D^{\frac{5}{2}} \otimes D^{1} \otimes D^{1}$ | 54 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{5}{2}} \otimes D^{1}$ | 72 |
| $D^{\frac{1}{2}} \otimes D^{19}$ | 78 | $D^{\frac{11}{2}} \otimes D^{1}$ | 36 | $D^{\frac{5}{2}} \otimes D^{1} \otimes D^{2}$ | 90 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{7}{2}} \otimes D^{1}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{20}$ | 82 | $D^{\frac{11}{2}} \otimes D^{2}$ | 60 | $D^{\frac{7}{2}} \otimes D^{1} \otimes D^{1}$ | 72 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{3}{2}} \otimes D^{1}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{21}$ | 86 | $D^{\frac{11}{2}} \otimes D^{3}$ | 84 | $D^{\frac{9}{2}} \otimes D^{1} \otimes D^{1}$ | 90 | $\bigotimes^{5} D^{\frac{1}{2}}$ | 32 |
| $D^{\frac{1}{2}} \otimes D^{22}$ | 90 | $D^{\frac{13}{2}} \otimes D^{1}$ | 42 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{1}{2}}$ | 8 | $\bigotimes^{4} D^{\frac{1}{2}} \otimes D^{\frac{3}{2}}$ | 64 |
| $D^{\frac{1}{2}} \otimes D^{23}$ | 94 | $D^{\frac{13}{2}} \otimes D^{2}$ | 70 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{3}{2}}$ | 16 | $\bigotimes^{4} D^{\frac{1}{2}} \otimes D^{\frac{5}{2}}$ | 96 |
| $D^{\frac{1}{2}} \otimes D^{24}$ | 98 | $D^{\frac{13}{2}} \otimes D^{3}$ | 98 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{5}{2}}$ | 24 | $\bigotimes^{3} D^{\frac{1}{2}} \otimes D^{1} \otimes D^{1}$ | 72 |
| $D^{\frac{3}{2}} \otimes D^{1}$ | 12 | $D^{\frac{15}{2}} \otimes D^{1}$ | 48 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{7}{2}}$ | 32 | $\bigotimes^{5} D^{\frac{1}{2}} \otimes D^{2}$ | 96 |
| $D^{\frac{3}{2}} \otimes D^{2}$ | 20 | $D^{\frac{15}{2}} \otimes D^{2}$ | 80 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{9}{2}}$ | 40 | | |
| $D^{\frac{3}{2}} \otimes D^{3}$ | 28 | $D^{\frac{17}{2}} \otimes D^{1}$ | 54 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{11}{2}}$ | 48 | | |
| $D^{\frac{3}{2}} \otimes D^{4}$ | 36 | $D^{\frac{17}{2}} \otimes D^{2}$ | 90 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{13}{2}}$ | 56 | | |
| $D^{\frac{3}{2}} \otimes D^{5}$ | 44 | $D^{\frac{19}{2}} \otimes D^{1}$ | 60 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{15}{2}}$ | 64 | | |
| $D^{\frac{3}{2}} \otimes D^{6}$ | 52 | $D^{\frac{19}{2}} \otimes D^{2}$ | 100 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{17}{2}}$ | 72 | | |
| $D^{\frac{3}{2}} \otimes D^{7}$ | 60 | $D^{\frac{21}{2}} \otimes D^{1}$ | 66 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{19}{2}}$ | 80 | | |
| $D^{\frac{3}{2}} \otimes D^{8}$ | 68 | $D^{\frac{23}{2}} \otimes D^{1}$ | 72 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{21}{2}}$ | 88 | | |
| $D^{\frac{3}{2}} \otimes D^{9}$ | 76 | $D^{\frac{25}{2}} \otimes D^{1}$ | 78 | $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \otimes D^{\frac{23}{2}}$ | 96 | | |
| $D^{\frac{3}{2}} \otimes D^{10}$ | 84 | $D^{\frac{27}{2}} \otimes D^{1}$ | 84 | $D^{\frac{1}{2}} \otimes D^{\frac{3}{2}} \otimes D^{\frac{3}{2}}$ | 32 | | |