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General Relativity with Electromagnetism in the Role of Gravity

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Abstract

This paper proves that electromagnetic fields as produced by charges, in analogy with gravitational fields as produced by energies, cause space-time curvatures, not because of the energy contents of the fields but because of the Coulomb potential of the charges; as a result, we have derived a special constant of proportionality between an electromagnetic energy-momentum tensor and Einstein tensor. The geodesics of the resultant electromagnetic 4-manifold represent the same dynamics as that given by the classical Lagrangian resulting in the Lorentz force law of motion.

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1 Introduction

This paper derives Einstein Field Equations ("EFE") for the classical electromagnetism of Maxwell and Poynting. In analogy with

$$R_{\mu\nu,grav} - \frac{1}{2}R_{grav}g_{\mu\nu,grav} = -\frac{8\pi G}{c^2}T_{\mu\nu,grav} \tag{1}$$

for gravitation by Einstein, we deduce

$$R_{\mu\nu,em} - \frac{1}{2}R_{em} \cdot g^{att;rep}_{\mu\nu,em} = -\frac{16\pi G}{\left(1 - \gamma^{-2}_{grav}g_{11,grav}\right)c^5}T^{att;rep}_{\mu\nu,em} \tag{2}$$

for electromagnetism. Following Einstein, this treatise makes use of the differential geometric property of Einstein tensor

$$\mathcal{E}_{\mu\upsilon} := R_{\mu\nu} - \frac{1}{2} R \cdot g_{\mu\nu} \tag{3}$$

being proportional to energy-momentum tensor $T_{\mu\nu}$ (cf. [3], 858) and applies weak field approximations (see [1], 814-818) to establish the constant of proportionality κ as based on weakly attractive or repulsive electromagnetic fields (cf. [9], 151-157 for a derivation of EFE). As such, there will be numerous "approximately-equal" signs in our derivation of κ.

The significance of our results is that, like gravity, the distribution of electric charges in space-time results in a 4-manifold $\mathcal{M}^4_{em}$ of curvatures and charges move along geodesics of $\mathcal{M}^4_{em}$, i.e., a geometrization of the electromagnetic force, which is a step toward a unified field theory (for some of the latest many attempts, see, e.g., [5, 8]; for related work integrating electromagnetism with EFE, cf. e.g., [6, 7]).

Section 2 below will first aim at deriving g_{em} (proving that the associated geodesics are exactly the classical electromagnetic Lagrangian), then $\mathcal{E}_{em}$, and finally

$$\frac{\mathcal{E}_{12,em}}{\mathcal{E}^{att;rep}_{11,em}} = \frac{-\left\|\bar{\mathbf{g}}\right\| \mathbf{V}_{Q,x}}{\pm\left\|\bar{\mathbf{S}}\right\|} \equiv \frac{T_{12,em}\ (\text{momentum})}{T^{att;rep}_{11,em}\ (\text{energy})}, \tag{4}$$

to obtain

$$\kappa_{em} = \frac{\mathcal{E}_{11}}{T_{11}}. \tag{5}$$

Section 3 will present a summary.

2 The Derivation

Definition 1 *The Minkowski space*

$$\mathbb{R}^{1+3} \quad : \quad = \{ (t, \mathbf{x} \equiv (x, y, z)) \in \mathbb{R}^4 \mid \textit{the inner product} \tag{6}$$

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle \quad : \quad = \mathbf{e}_i^T \eta \mathbf{e}_j,\ i, j = 1, 2, 3, 4, \tag{7}$$

$$\eta \quad : \quad = \textit{diag}\left(1, -c^{-2}, -c^{-2}, -c^{-2}\right)_{\mathbf{E}}\ , \tag{8}$$

$$\mathbf{E} \quad \equiv \quad \left(\mathbf{e}_i \equiv (\textit{Kronecker}\ \delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4})\right)^4_{i=1}\ , \tag{9}$$

$$c \quad \equiv \quad \textit{the speed of light in the vacuum}\}. \tag{10}$$

Let $f : U_{(0,\mathbf{0})} \subset \mathbb{R}^{1+3} \longrightarrow \mathcal{M}^4$ *be a local parametrization of the space-time 4-manifold* $\mathcal{M}^4$*; we call* U *the laboratory frame* S *and the coordinates of the* $t-$*axis the proper times of* S*, i.e.,*

$$t_o(S) \equiv (t_o, 0, 0, 0) \in U. \tag{11}$$

Remark 1 *If $\mathcal{M}^4 = \mathbb{R}^{1+3}$, then $f =$ the Lorentz transformation $\mathcal{L}$; $\mathcal{L} : S \longrightarrow \tilde{S}$ has the following matrix representation if $(t,x,y,z) = (0,0,0,0) = \left(\tilde{t},\tilde{x},\tilde{y},\tilde{z}\right)$ and $\mathcal{L}(1,V,0,0) = \left(\tilde{t}_o,0,0,0\right)$:*

$$L = \gamma \begin{pmatrix} 1 & -\frac{V}{c^2} \\ -V & 1 \end{pmatrix}_{(\mathbf{e}_1,\mathbf{e}_2)}, \tag{12}$$

where $(V,0,0)$ is the velocity of $\tilde{S}$ relative to S and

$$\gamma \equiv \left(1 - \left(\frac{V}{c}\right)^2\right)^{-\frac{1}{2}} \in [1,\infty) \tag{13}$$

is the Lorentz factor. Consider an emission of light at $t_o = 0 = \tilde{t}_o$ in the direction of $V \in \mathbb{R}$; then $\forall t_o, \tilde{t}_o > 0$ S observes $(t_o, t_o c)$ and $\tilde{S}$ observes $\left(\tilde{t}_o, \tilde{t}_o c\right)$; further,

$$L\left(t_o, t_o c\right)^T = \gamma\left(1 - \frac{V}{c}\right) \cdot \left(t_o, t_o c\right)^T = \left(\tilde{t}_o, \tilde{t}_o c\right)^T; \tag{14}$$

thus,

$$\frac{\tilde{t}_o}{t_o} = \gamma\left(1 - \frac{V}{c}\right) = \lambda, \text{ an eigenvalue of } L. \tag{15}$$

Note that

$$\gamma\left(1 - \frac{V}{c}\right) \cdot \gamma\left(1 + \frac{V}{c}\right) = 1; \tag{16}$$

i.e., L has two eigenvalues

$$\lambda_{\max} = \gamma\left(1 + \frac{|V|}{c}\right) > 1, \text{ and} \tag{17}$$

$$\lambda_{\min} = \gamma\left(1 - \frac{|V|}{c}\right) < 1. \tag{18}$$

Remark 2 *In the above, if $V = 0$, then $\mathcal{L} = I$; consider now $V(t) \equiv 0$ $\forall t \in (-\infty, 0]$ but*

$$\forall t \in (0,T] \quad V(t) \approx at \tag{19}$$

for some $T > 0$ and some constant acceleration $a > 0$, due to the existence of some force. Then

$$\lambda = \frac{\tilde{t}_o}{t_o} = \gamma(t)\left(1 - \frac{V(t)}{c}\right) \tag{20}$$

measures the curvatures of $\mathcal{M}^4$ over $(0,T]$; this treatment of λ will be assumed hereafter.

Since $V(t) > 0$ *on* $(0, T]$, *we have*

$$\lambda = \sqrt{\frac{c - V(t)}{c + V(t)}} < 1; \tag{21}$$

by well established observations such as particles of fleeting existence can nevertheless gravitate from space to Earth to be observed, we deduce that $\lambda < 1$ *for attractive forces; by a reversal of time in the preceding dynamics, we deduce that* $\lambda > 1$ *for repulsive forces. We will thus make the following distinction and notation:*

$$\lambda_{att} \; : \; = \gamma \left(1 - \frac{|V|}{c}\right) < 1, \text{ and} \tag{22}$$

$$\lambda_{rep} \; : \; = \gamma \left(1 + \frac{|V|}{c}\right) > 1. \tag{23}$$

Further, note that $\forall \left(\frac{V}{c}\right) \approx 0$, *one uses*

$$\frac{m_o}{\lambda_{att}} \approx m_o \gamma \text{ and} \tag{24}$$

$$\frac{m_o}{\lambda_{rep}} \approx m_o \gamma^{-1} \tag{25}$$

for (Special) relativistic adjustment of a mass. Also, a metric g *on* $\mathcal{M}^4$ *by definition is such that*

$$g_{11} = \left(\frac{\tilde{t}_o}{t_o}\right)^2 = (\lambda_{att;\, rep})^2 \approx \lambda_{att}^{\pm 2}. \tag{26}$$

Remark 3 *Let* $p_1, p_2 \in \mathcal{M}^4$; *then a maximization of*

$$\int_{f^{-1}(p_1)}^{f^{-1}(p_2)} \frac{d\tilde{t}_o}{dt_o} dt_o \tag{27}$$

over all trajectories $\{(t, x(t), y(t), z(t))\}$ *derives the geodesic from* p_1 *to* p_2 *maximizing the proper time elapsed in* $\tilde{S}$.

Proposition 1 *Let* g *be a local metric of* $\mathcal{M}^4$ *and express* g *as a matrix in the basis of* $\mathbf{B} \equiv \left\{\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\}$; *if* $f \approx \mathcal{L}$ *(i.e.,* $\mathcal{M}^4$ *is near flat), then*

$$\frac{d\tilde{t}_o}{dt_o} = (1, 0, 0, 0) \; g_{\mathbf{B}} \; (\mp 1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z)^T. \tag{28}$$

Proof. Without loss of generality, consider

$$L = \gamma \begin{pmatrix} 1 & \pm\frac{\mathbf{V}}{c^2} \\ \pm\mathbf{V} & 1 \end{pmatrix} \tag{29}$$

and calculate $(1,0)\ g_{\mathbf{B}}\ (\mp 1, \mathbf{V})$

$$= \ (1,0)\left(\left(L^{-1}\right)^T\right)^{-1}\left[\left(L^{-1}\right)^T g_{\mathbf{B}}\ L^{-1}\right]\ L\,(\mp 1,\mathbf{V})^T \tag{30}$$

$$\approx\ (1,0)\left(\gamma\begin{pmatrix} 1 & \pm\mathbf{V} \\ \pm\frac{\mathbf{V}}{c^2} & 1\end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{c^2}\end{pmatrix}\begin{pmatrix}\mp\gamma^{-1} \\ 0\end{pmatrix} \tag{31}$$

$$= \ (\gamma, \pm\gamma\mathbf{V})\begin{pmatrix}\Delta\tilde{t}_o \\ 0\end{pmatrix} \text{ (observe that } L : (\mp 1,\mathbf{V})^T \longmapsto \left(\Delta\tilde{t}_o, 0\right)^T, \tag{32}$$

where $\Delta\tilde{t}_o$ equal to $\mp\gamma^{-1}$ is the proper time of $\tilde{S}$ by definition)

$$= \ \frac{\Delta\tilde{t}_o}{\sqrt{1-\left(\frac{\mathbf{V}}{c}\right)^2}} = \frac{\Delta\tilde{t}_o}{\left\|(\mp 1, -\mathbf{V})^T\right\|_\eta} = \frac{\Delta\tilde{t}_o}{\left\|L^{-1}(\mp 1, -\mathbf{V})^T\right\|_\eta} \tag{33}$$

$$= \ \frac{\Delta\tilde{t}_o}{(\mp\gamma^{-1}, 0)} = \frac{\Delta\tilde{t}_o}{\Delta t_o} \approx \frac{d\tilde{t}_o}{dt_o}, \text{ (where } L^{-1} : (\mp 1, -\mathbf{V})^T \longmapsto (\Delta t_o, 0)^T, \tag{34}$$

analogous to the above Equation (32)).

■

The Setup - -

We consider the dynamics of a charge Q at $(0,0,0,0) \in U$ that attracts or repels a charge q at $(0,x,y,z) \in U$, where

$$r_\infty \equiv \sqrt{(x^2+y^2+z^2)} \text{ is such that } r_\infty^{-1} \approx 0. \tag{35}$$

Theorem 1 *(Feynman [2], II-28-2) The field momentum produced by Q is*

$$\mathbf{P}(t) = \frac{Q^2}{4\pi\epsilon_o r_o c^2}\mathbf{V}_Q(t), \tag{36}$$

where $\epsilon_o \equiv$ the permittivity constant $\approx \frac{1}{9\times 4\pi} \times 10^{-9} \times \frac{coulomb^2\cdot second^2}{kilogram\cdot meter^3}$, $r_o \equiv$ the "classical electron radius" $\approx 2.82 \times 10^{-15}$ meter, and $\mathbf{V}_Q(t) << c$ is the velocity of Q at t.

Remark 4 *We note that the above Equation* (36) *was derived in* [2] *by an integration over the (continuous) field energy densities (cf.* [2], *II-28-2 and II-8-12). Thus, to apply Equation* (36) *to the above* Setup *of exactly two (discrete) point charges, we must have*

$$Q = q = \text{ the smallest charge } = \text{ an electron.} \tag{37}$$

Definition 2

$$\textit{The average field momentum density } \bar{\mathbf{g}}(t) := \mathbf{P}(t) / \left(\frac{4\pi r_\infty^3}{3}\right). \tag{38}$$

Theorem 2 *(Feynman [2], II-27-9) The Poynting vector* $\mathbf{S}$ *is related to the momentum density* $\mathbf{g}$ *by*

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}. \tag{39}$$

Corollary 1

$$\mathbf{P}(t) = \left(\frac{4\pi r_\infty^3}{3}\right)\bar{\mathbf{g}}(t) \tag{40}$$

$$= \left(\frac{4\pi r_\infty^3}{3}\right)\frac{\bar{\mathbf{S}}(t)}{c^2}. \tag{41}$$

$$\textit{where } \bar{\mathbf{S}}(t) \equiv \textit{ the average field energy flow in the direction} \tag{42}$$

$$\textit{of } \mathbf{V}_Q(t), \textit{ with unit equal to } \left(\frac{joule}{second \cdot meter^2}\right). \tag{43}$$

Theorem 3 *(Feynman [2], II-27-11: Conservation of the total momentum of particles and field)*

$$\mathbf{P}(t) \equiv m_{Q,o}\mathbf{V}_Q(t) = -m_{q,o}\mathbf{V}(t), \tag{44}$$

where $m_{Q,o}$ *and* $m_{q,o}$ *are respectively the rest masses of* Q *and* q.

Remark 5 *For the above theorem, the laboratory frame* S *is set to move at the constant velocity* $\bar{\mathbf{V}}$ *so that* $m_{Q,o}\mathbf{V}_Q(t) + m_{q,o}\mathbf{V}(t) = \mathbf{0}$, *where*

$$\bar{\mathbf{V}} = \frac{m_{Q,o}\mathbf{V}_Q(t) + m_{q,o}\mathbf{V}(t)}{m_{Q,o} + m_{q,o}}. \tag{45}$$

Remark 6 *The Newton's law of motion as adjusted for the effect of Special Relativity is*

$$\mathbf{F}^{att;\ rep} = \left(\gamma^{\pm 1} m_o\right)\left(\gamma^{\pm 2}\mathbf{a}\right) \tag{46}$$

respectively for attractive and repulsive force $\mathbf{F}^{att;\ rep}$ *if* $\mathbf{a}$ *is in the direction of* $\mathbf{V}$ *(cf.* [4], *Equation* (13.31), *272-273; also, Equations* (24),(25) *above).*

Proposition 2 *Let* $v(t) := \|\mathbf{V}(t)\|$ *and* $v_Q(t) := \|\mathbf{V}_Q(t)\|$*; then*

$$\gamma^{\pm 2}\left(\frac{v(t)}{c}\right) = \frac{\text{the electric potential energy } PE_e \text{ of } Q \text{ and } q}{\text{the rest energy } RE \text{ of } q}. \tag{47}$$

Proof. By Theorems 1 and 3,

$$\left(\frac{v(t)}{c}\right) = \left(\frac{1}{m_{q,o}c^2}\right)\cdot q\left(\frac{Q}{q}\frac{v_Q(t)}{c}\frac{r_\infty}{r_o}\right)\cdot\frac{Q}{4\pi\epsilon_o r_\infty} \tag{48}$$

$$\equiv \frac{1}{RE}\cdot K\cdot\frac{qQ}{4\pi\epsilon_o r_\infty}, \tag{49}$$

where

$$K \equiv \frac{Q}{q}\frac{v_Q(t)}{c}\frac{r_\infty}{r_o} = \frac{v_Q(t)\cdot\left(\frac{r_\infty}{c}\right)}{r_o} \text{ (cf. Remark 4)} \tag{50}$$

is an electrodynamic adjustment factor of the electrostatic potential (cf. [2], II-15-14, 15);

$$K = 1 \text{ if } v_Q(t)\cdot\left(\frac{r_\infty}{c}\right) \approx v_Q\cdot\left(\frac{r_\infty}{c}\right) = r_o, \tag{51}$$

i.e., the point charge Q travels to the boundary of the "classical electron," or equivalently, Q is a stationary electron. Thus, taking into account the effect of Special Relativity, we have

$$\gamma^{\pm 2}\left(\frac{v(t)}{c}\right) = \frac{\gamma^{\pm 2}KQq}{RE} = \frac{PE_e}{RE}. \tag{52}$$

■

Corollary 2

$$-\gamma^{\pm 2}\left(\frac{v(t)}{c}\right)\left(\frac{v(t)\,v_Q(t)}{c^2}\right) = \frac{q\mathbf{V}(t)\cdot\mathbb{A}(t)}{RE}, \tag{53}$$

where $\mathbb{A}(t) :=$ *the vector potential, or curl* $\mathbb{A}(t) =$ *the magnetic field* $\mathbb{B}$.

Proof. Since

$$-v(t)\,v_Q(t) = \mathbf{V}(t)\cdot\mathbf{V}_Q(t) \text{ and} \tag{54}$$

$$\frac{\gamma^{\pm 2}KQ\mathbf{V}_Q(t)}{4\pi\epsilon_o r_\infty c^2} = \mathbb{A}(t) \text{ ([2], II-14-4)}, \tag{55}$$

we have

$$-\gamma^{\pm 2}\left(\frac{v(t)}{c}\right)\left(\frac{v(t)\,v_Q(t)}{c^2}\right) \tag{56}$$

$$= \frac{\gamma^{\pm 2}KQ\; q\mathbf{V}(t)\cdot\mathbf{V}_Q(t)}{RE\cdot 4\pi\epsilon_o r_\infty c^2} = \frac{q\mathbf{V}(t)\cdot\mathbb{A}(t)}{RE}. \tag{57}$$

■

Definition 3 *We call an electromagnetic field attractive if the total potential energy is negative, and repulsive if the total potential energy is positive.*

Proposition 3 *For any weakly attractive or repulsive electromagnetic field, the metric $g_{em}^{att;\ rep}$ has the following matrix representation in the basis of* **B** *(refer to Proposition 1 above):*

$$g_{em}^{att;\ rep} = \begin{pmatrix} \lambda_{em}^{\pm 2} & -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_x}{c^3} & -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_y}{c^3} & -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_z}{c^3} \\ -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_x}{c^3} & o\left(\frac{v}{c}\right) - c^{-2} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)^3 \\ -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_y}{c^3} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right) - c^{-2} & o\left(\frac{v}{c}\right)^3 \\ -\frac{2\gamma^{\pm 2} v_Q \mathbf{V}_z}{c^3} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right) - c^{-2} \end{pmatrix}. \quad (58)$$

Proof. First, we note that besides being symmetric, $g_{em}^{att;\ rep} \longrightarrow \eta$, as $\mathbf{V}, \mathbf{V}_Q \longrightarrow \mathbf{0}$. Second,

$$g_{11,em}^{att;\ rep} = \lambda_{em}^{\pm 2} = \left(\frac{\tilde{t}_o}{t_o}\right)^2_{att;\ rep} \text{ (cf. Equation (26)).} \quad (59)$$

Third, by Proposition 1 we have

$$\begin{aligned} \frac{d\tilde{t}_o}{dt_o} &= (1,0,0,0)\ g_{\mathbf{B}}\ (\mp 1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z)^T && (60) \\ &= \mp\lambda^{\pm 2} - \frac{2\gamma^{\pm 2} v_Q v^2}{c^3} && (61) \\ &\approx \mp\gamma^{\pm 2}\left(1 \mp \frac{2v}{c}\right) + \frac{2q\mathbf{V}\cdot\mathbb{A}}{RE} \text{ (by Corollary 2)} && (62) \\ &\approx \mp\left(1 \pm \left(\frac{v}{c}\right)^2\right) + \frac{2\left(PE_e + q\mathbf{V}\cdot\mathbb{A}\right)}{RE} \text{ (by Proposition 2)} && (63) \\ &= \mp 1 - \frac{m_o v^2}{m_o c^2} + \frac{2\left(PE_e + q\mathbf{V}\cdot\mathbb{A}\right)}{RE} && (64) \\ &= \mp 1 - \frac{2\left(\text{kinetic energy } KE - PE_e - q\mathbf{V}\cdot\mathbb{A}\right)}{RE}, && (65) \end{aligned}$$

which is equivalent to Feynman's least action for the classical electrodynamics ([2], II-19-7). ■

Corollary 3 *The Einstein tensor*

$$\mathcal{E}_{em}^{att;\ rep} \approx \begin{pmatrix} \mp\frac{6v}{r_k^2 c} & -\frac{6v_Q\mathbf{V}_x}{r_k^2 c^3} & -\frac{6v_Q\mathbf{V}_y}{r_k^2 c^3} & -\frac{6v_Q\mathbf{V}_z}{r_k^2 c^3} \\ -\frac{6v_Q\mathbf{V}_x}{r_k^2 c^3} & -O\left(r_k^{-2}\right) & O\left(r_k^{-2}c^{-4}\right) & O\left(r_k^{-2}c^{-4}\right) \\ -\frac{6v_Q\mathbf{V}_y}{r_k^2 c^3} & O\left(r_k^{-2}c^{-4}\right) & -O\left(r_k^{-2}\right) & O\left(r_k^{-2}c^{-4}\right) \\ -\frac{6v_Q\mathbf{V}_z}{r_k^2 c^3} & O\left(r_k^{-2}c^{-4}\right) & O\left(r_k^{-2}c^{-4}\right) & -O\left(r_k^{-2}\right) \end{pmatrix}_{\mathbf{B}}. \quad (66)$$

Proof. $\mathcal{E}_{\mu\upsilon} := R_{\mu\nu} - \frac{1}{2}R \cdot g_{\mu\nu}$; $\forall \mathcal{M}^4 \approx \mathbb{R}^{1+3}$ we have

$$(R_{\mu\nu}) \approx diag\left(-\frac{3}{r_K^2}, -\frac{1}{r_K^2}, -\frac{1}{r_K^2}, -\frac{1}{r_K^2}\right) \text{ and} \tag{67}$$

$$R \approx -\frac{6}{r_K^2}, \tag{68}$$

where $r_K \equiv$ the radius of sectional curvatures (cf. [3], 860; [9], 154). Thus, substituting Equation (58) into $(g_{\mu\nu})$ in $(\mathcal{E}_{\mu\upsilon})$, we arrive at the conclusion. ■

Lemma 4 *Let*

$$\bar{m}_{q,o} \equiv \frac{m_{q,o}}{(4\pi r_\infty^3/3)}; \tag{69}$$

then

$$\bar{m}_{q,o} r_\infty^2 \approx \left(1 - \gamma_{grav}^{-2} g_{11,grav}\right) \cdot \frac{3c^2}{8\pi G}, \tag{70}$$

where

$$g_{11,grav} \approx \lambda_{grav}^2 \approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{V}_\alpha}{c}\right), \tag{71}$$

with $\mathbf{V}_\alpha$ $\equiv$ *the radial velocity* (> 0) *of any arbitrary particle* α *gravitating toward* q *at a distance of* r_∞*, and* $G \equiv$ *the universal gravitational constant.*

Proof.

$$g_{11,grav} \approx \lambda_{grav}^2 \approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{V}_\alpha}{c}\right) \text{ (refer to Equation (26))} \tag{72}$$

$$\approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{a}_\alpha t}{c}\right) \text{ (cf. Remark 2)} \tag{73}$$

$$= \gamma_{grav}^2 \left(1 - \frac{2G\bar{m}_{q,o}}{r_\infty^2 c} \cdot \frac{4\pi r_\infty^3}{3} \cdot \frac{r_\infty}{c}\right); \tag{74}$$

thus,

$$\bar{m}_{q,o} r_\infty^2 \approx \left(1 - \gamma_{grav}^{-2} g_{11,grav}\right) \cdot \frac{3c^2}{8\pi G}. \tag{75}$$

■

Remark 7 *The above lemma expresses the gravitating mass density of* q *in terms of its effect on* $\mathcal{M}^4$ *as measured by* $g_{11,grav}$*; by the principle of equivalence,* $\bar{m}_{q,o}$ *is also the inertial mass density, and in the next theorem* $\bar{m}_{q,o}$ *is to be treated as such. Also, note that as* $r_\infty^{-1} \longrightarrow 0$*, we have* $\left|r_\infty^{-2} - r_K^{-2}\right| \longrightarrow 0$.

Theorem 5

$$\mathcal{E}_{\mu\nu,em}^{att;\ rep} := R_{\mu\nu,em} - \frac{1}{2} R_{em} \cdot g_{\mu\nu,em}^{att;\ rep} = -\frac{16\pi G}{\left(1-\gamma_{grav}^{-2} \cdot g_{11,grav}\right) c^5} T_{\mu\nu,em}^{att;\ rep}. \quad (76)$$

Proof.

$$\frac{\mathcal{E}_{12,em}}{\mathcal{E}_{11,em}^{att;\ rep}} = \pm\frac{1}{c^2}\left(\frac{v_Q}{v}\right)\mathbf{V}_x \text{ (by Equation (66))} \quad (77)$$

$$= \pm\frac{1}{c^2} \cdot \left(\frac{m_{q,o}}{m_{Q,o}}\right) \cdot \left(-\frac{m_{Q,o}}{m_{q,o}}\mathbf{V}_{Q,x}\right) \text{ (by Equation (44))} \quad (78)$$

$$= \frac{-\frac{\|\bar{\mathbf{s}}\|}{c^2}\mathbf{V}_{Q,x}}{\pm\|\bar{\mathbf{S}}\|} = \frac{-\|\bar{\mathbf{g}}\|\,\mathbf{V}_{Q,x}}{\pm\|\bar{\mathbf{S}}\|} \text{ (by Equation (41))} \quad (79)$$

$$\equiv \frac{T_{12,em}}{T_{11,em}^{att;\ rep}}, \quad (80)$$

where $T_{11,em}^{att;\ rep}$ and $T_{1j,em},\ j=2,3,4$, are respectively the energy-flow and the momentum densities. Thus,

$$\mathcal{E}_{em}^{att;\ rep} = \kappa_{em} T_{em}^{att;\ rep} \text{ has} \quad (81)$$

$$\kappa_{em} = \frac{\mathcal{E}_{11,em}^{att;\ rep}}{T_{11,em}^{att;\ rep}} = \mp\frac{6v}{r_k^2 c} / \pm\|\bar{\mathbf{S}}\| \text{ (by Equations (66), (80))}, \quad (82)$$

but

$$\|\bar{\mathbf{S}}\| = \frac{3c^2}{4\pi r_\infty^3} \cdot m_{q,o} v \text{ (by Equations (41), (44))}, \quad (83)$$

so

$$\kappa_{em} = -\frac{6}{r_K^2 c} \cdot \frac{4\pi r_\infty^3}{3c^2 m_{q,o}} \quad (84)$$

$$= -\frac{6}{r_\infty^2 c} \cdot \frac{1}{c^2 \bar{m}_{q,o}} \text{ (cf. Remark 7)} \quad (85)$$

$$= -\frac{6}{c^3} \cdot \frac{8\pi G}{\left(1-\gamma_{grav}^{-2} g_{11,grav}\right) \cdot 3c^2} \text{ (by the above Lemma)} \quad (86)$$

$$= -\frac{16\pi G}{\left(1-\gamma_{grav}^{-2} \cdot g_{11,grav}\right) c^5}. \quad (87)$$

■

Remark 8 $T_{11,em}^{att;\ rep} \equiv \pm\|\bar{\mathbf{S}}\|$ *has unit (recalling from Equation (43))*

$$\frac{joule}{second \cdot meter^2} \quad (88)$$

$$= \frac{kilogram \cdot meter^2}{second^2} \cdot \frac{1}{second \cdot meter^2} \quad (89)$$

$$= \frac{kilogram}{second^3}, \quad (90)$$

so that $\left(\kappa_{em} \cdot T^{att;rep}_{11,em}\right)$ *has unit*

$$= \frac{[G]}{[c^5]} \cdot \frac{kilogram}{second^3} \tag{91}$$

$$= \frac{meter^3}{kilogram \cdot second^2} \cdot \frac{second^5}{meter^5} \cdot \frac{kilogram}{second^3} \tag{92}$$

$$= \frac{1}{meter^2} = \left[\frac{1}{r_k^2}\right], \tag{93}$$

measuring the local curvatures of $\mathcal{M}^4_{em}$. *We emphasize that our* $T_{11,em}$ *represents energy flows in a specific direction across an area of square meter per second, which is different from the common identification of* $T_{11,em}$ *with stationary energy densities with unit:* $[joule/(meter^3)]$ *(see, e.g.,* [9], *45, equation* (2.8.10)*).*

3 Summary

As Feynman indicated ([2], II-19-8,9), the least action in quantum electrodynamics is the same as that of the classical; in this paper we have shown that the classical least action is a geodesic of our $\mathcal{M}^4_{em}$; thus, we have contributed a geometric underpinning of both the classical and quantum electrodynamics.

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