Adv. Studies Theor. Phys., Vol. 1, 2007, no. 8, 405 - 408

Mass-Sign Duality of Cubic Oscillators

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Abstract

In the light of the guiding methodical role of anharmonic oscillators in field theory we communicate the observation that their "first non-trivial" $\mathcal{PT}-$symmetric cubic Hamiltonians $H_{\pm} = p^2 \pm m^2 x^2 + \mathrm{i}\, f\, x^3$ with opposite sign of mass term are, up to a constant shift, isospectral.

Quantitative description of quantum fields is mostly based on perturbation theory where a free Lagrangean, say,

$$\mathcal{L}_{\rm free} = \frac{1}{2}\, [\partial\phi(z)]^2 - \frac{1}{2}\, m^2\, [\phi(z)]^2$$

is being complemented by a suitable interaction term $\mathcal{L}_{\rm int}$ proportional to a "small" coupling constant λ. On a formal level, an enormous popularity of the similar models with polynomial $\mathcal{L}_{\rm int} \sim [\phi(z)]^3 + \ldots$ is based on their renormalizability as well as on the feasibility of the comparatively straightforward manipulations with Feynman diagrams. Often, a useful methodical guide may also be provided by analogies between the quantized field in three dimensions and its "zero-dimensional" formal analogues appearing in quantum mechanics. In suitable units the Lagrangeans are then replaced by Hamiltonians,

$$H = H(\lambda) = -\frac{d^2}{dx^2} + m^2 x^2 + \lambda\, \left(g_3\, x^3 + g_4\, x^4 + \ldots\right), \qquad |\lambda| \ll 1$$

and the spectra and/or bound eigenstates are sought in the form of the Rayleigh-Schrödinger asymptotic series [1],

$$E = E_n(\lambda) = (2n+1)\sqrt{m^2} + \lambda\, E_n^{[1]} + \lambda^2\, E_n^{[2]} + \ldots .$$

Perturbation studies of various anharmonicities proved enormously rewarding and many convergence questions emerging within the formalism have been clarified. These studies also helped to reveal several unexpected features of the models themselves. *Pars pro toto* let us mention that Bender and Wu [2] revealed that for the quartic anharmonic oscillator with $H_{\rm int} = \lambda\, g_4\, x^4$, *all* the bound-state energies $E_n(\lambda)$ with $n = 0, 1, \ldots$ coincide with the values of a *single* analytic multivalued function, considered simply on its *separate* Riemann sheets.

The validity of similar observations has been confirmed for a few other types of interaction terms. During their studies an amazing empirical observation has been made in the case of the "first nontrivial" cubic anharmonicity $H_{\rm int} = \lambda\, g_3\, x^3$. An analytic continuation of its energies $E_n(\lambda)$ to the purely imaginary coupling range $\lambda\, g_3 \equiv {\rm i}\, f$ with real f (where the Hamiltonian itself remains manifestly non-Hermitian, $H \neq H^\dagger$) revealed that *all* the spectrum seems to remain *real*, $E_n(\lambda) \in I\!\!R$. After an original, purely perturbative support of such an unexpected possibility given in ref. [3], decisive WKB and numerical demonstrations of its plausibility have been presented in refs. [4] and [5]. Finally, rigorous proof of the reality of the whole spectrum has been delivered in refs. [6] and [7].

In what follows we intend to point out that the list of the exciting features of the Hamiltonians

$$H = H(m^2, f) = -\frac{d^2}{dx^2} + m^2 x^2 + {\rm i}\, f\, x^3$$

may be complemented by an observation concerning an interesting nonperturbative symmetry aspect of this model.

Theorem.

Hamiltonians $H_+ = H(m^2, f) + 2 f^{-2}(m^2/3)^3$ and $H_- = H(-m^2, f) + 2 f^{-2}(-m^2/3)^3$ are isospectral.

Proof.

The real-mass anharmonic-oscillator Schrödinger equation

$$-\frac{d^2}{dx^2}\,\psi(x) + m^2 x^2\,\psi(x) + {\rm i}\, f\, x^3\,\psi(x) = E_{[RM]}\,\psi(x)\,, \qquad \psi(x) \in L_2(-\infty, \infty) \tag{1}$$

defines wave functions which are analytic in the whole complex plane. By construction they asymptotically vanish not only along the right and left half-axes but rather within the whole neighboring wedges, i.e., along all the half-lines at angles $\alpha \in (-3\pi/10, \pi/10)$ (forming the right wedge) and, similarly, within the left wedge with $\alpha \in (9\pi/10, 13\pi/10)$.

For this reason all the wave functions may be analytically continued to a shifted integration contour $x = T + {\rm i}\,\gamma$ at any real constant γ. We may then define the new wave functions

$$\psi(T + \mathrm{i}\,\gamma) \equiv \phi(T) \in L_2(-\infty, \infty)$$

which satisfy the following "imaginary mass" Schrödinger equation

$$-\frac{d^2}{dT^2}\,\phi(T) - m^2T^2\,\phi(T) + \mathrm{i}\,f\,T^3\,\phi(T) = E_{[IM]}\,\phi(T)\,. \qquad (2)$$

The new and old energies are related by the identity

$$E_{[IM]} \;\equiv\; \left(E_{[RM]} + \frac{4\,m^6}{27\,f^2}\right)\,. \qquad (3)$$

This means that Schrödinger equations with potentials

$$V_{\pm}(x) = \frac{2\,\left(\pm m^2\right)^3}{27\,f^2} \pm m^2x^2 + \mathrm{i}\,f\,x^3 \qquad (4)$$

are isospectral. QED.

One of the puzzling consequences of the latter observation is that perturbation theory in its standard form employing the smallness of the coupling f can only be applied in the real-mass case. The behaviour of the isospectral partner (2) of the real-mass problem (1) becomes manifestly non-perturbative, characterized by the divergence of the energy shift in eq. (3).

Although this shift smoothly disappears in the vanishing-mass limit, it makes an impression of having a "wrong" sign at any $m^2 > 0$. This may be read as another paradox related to the nontriviality of the "correct" and positive definite scalar product which makes our Hamiltonian physical, i.e., (quasi-)Hermitian [8]. Indeed, only with respect to this nonstandard scalar product (discussed and even constructed by several authors [5]) one can understand that and how the transition to the "wrong sign" mass term in eq. (2) becomes *precisely* compensated by the shift of the spectrum via eq. (3).

Acknowledgement

Work supported by the Institutional Research Plan AV0Z10480505, by the MŠMT "Doppler Institute" project Nr. LC06002 and by GAČR, grant Nr. 202/07/1307.

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Received: June 15, 2007