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Spin 1/2 Bound States in Schwarzschild Geometry

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Abstract

The Dirac equation is separated, by separation of variables, in the Schwarzschild space-time. The radial equations are reduced to a pair of decoupled ordinary differential equations of the second order. Asymptotic behaviours of the radial solutions are determined. A covariant scalar product between states is considered that is induced by the conserved current. It is shown that a class of states exist that are bound with respect to the scalar product. The states characterized by asymptotic polynomial-like behaviour are associated to a Hydrogen-like energy spectrum.

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1 Introduction

The Schwarzschild space-time is a concrete example of curved space-time where the field equations can be separated (by variable separation) for arbitrary value of the spin [14]. The corresponding separated angular equations, that coincide with those relative to the Robertson-Walker space-time [13], can be disentangled in general. The regular solutions can be explicitly given, at least for the lowest values of the spin (see e.g., [11]). For what concerns the separated radial equations, they can be reduced to disentangled ordinary differential equations of the second order for the scalar and Dirac field. For spin 1, 3/2, and *a fortiori* for higher values of the spin, the radial equations can be disentangled, in principle, by an elementary substitution method, but they result in ordinary differential equations of fourth or higher order [14, 15].

There is a general difficulty (already present for the scalar field) to obtain the exact radial solutions for the fields equations in Schwarzschild space-time. In case of Dirac field, approximated solutions have been given [10, 6, 7]. Recently it has been put into evidence the existence of states that are bound in the (scalar) product induced by the conserved current. These states are associated to a continuum set of values of the energy and not to a discrete one as it happens in ordinary Quantum Mechanics. Their existence can be proved for spin 1, 3/2 [15, 16] and also holds for spin 0 field as it follows by applying the present study to the results of Ref. [12]. Unfortunately, the treatment of Ref. [10] is not suitable to decide whether that property holds also for spin 1/2.

In the present paper the Dirac equation in Schwarzschild space-time is reconsidered in the light of the above discussion. The equation is separated by the Newman-Penrose formalism [8] in the line of the original separation of the equation performed by Chandrasekhar in Kerr metric [2]. The angular equations admit exact regular solutions. The separated radial equations are reduced to a system of two differential equations of the second order. The solution of the radial equations are obtained locally at $r = 0$, $r = 2M$ and $r = \infty$. Even if an exact solution of the radial equation seems difficult to be obtained, some global aspects are given. They allow to show that states that are bound in the product induced by the conserved current do indeed exist for Dirac equation in Schwarzschild geometry. The bound states are labelled again by a continous parameter. Formally it is possible to select among the bound states those with an asymptotical polynomial-like behaviour. Under a constraint on physical parameters these kind of states result to be associated to a Hydrogen like energy spectrum. The existence of the discrete spectrum is here a formal aspects of the scheme that exists therefore for spin 0, 1/2, 1, 3/2. (for spin 1 see the similar but different scheme of Ref. [3]). The guess is that the property holds for every value of the spin.

2 Separation and Solution of the equation

The Dirac equation in curved space-time [4, 9] can be given in the form [2]

$$\begin{aligned}\nabla_{AX'}P^A + i\mu_\star\overline{Q}_{X'} &= 0\\ \nabla_{AX'}Q^A + i\mu_\star\overline{P}_{X'} &= 0\end{aligned} \tag{1}$$

$\nabla_{AX'}$ the covariant spinor derivative, $\mu_\star\sqrt{2} = m_e$ the mass of the particle. The equation can be explicited by the Newman-Penrose formalism in term of the (tabulated) spin coefficients and directional derivatives:

$$\begin{aligned}(D+\epsilon-\rho)P^0 + (\delta^\star+\pi-\alpha)P^1 - i\mu_\star\bar{Q}^1 &= 0\\ (\delta+\beta-\tau)P^0 + (\Delta+\mu-\gamma)P^1 + i\mu_\star\bar{Q}^0 &= 0\\ (D+\epsilon-\rho)Q^0 + (\delta^\star+\pi-\alpha)Q^1 - i\mu_\star\bar{P}^1 &= 0\\ (\delta+\beta-\tau)Q^0 + (\Delta+\mu-\gamma)Q^1 + i\mu_\star\bar{P}^0 &= 0\end{aligned} \tag{2}$$

In case of the Schwarzschild space-time of line element

$$ds^2 = \Big(1-\frac{2M}{r}\Big)dt^2 - \Big(1-\frac{2M}{r}\Big)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{3}$$

by adopting the null tetrad frame constructed in Ref. [2] the equation (2) can be separetd by variable separation by setting

$$\begin{aligned}(P_0,\, P_1) &\equiv \frac{1}{r}\Big(-H_2(r)S_2(\theta),\, H_1(r)S_1(\theta)\Big)\,e^{i\sigma t+im\varphi}\\ (\bar{Q}_0,\, \bar{Q}_1) &\equiv -\frac{1}{r}\Big(H_2(r)S_1(\theta),\, H_1(r)S_2(\theta)\Big)\,e^{i\sigma t+im\varphi}\end{aligned} \tag{4}$$

where one can assume $m = 0, \pm 1, \pm 2, \ldots$ and σ, a priori, a complex number.

The separated angular equations can be reduced to the eigenvalue problem

$$\begin{aligned} L^- L^+ S_1 &= -\lambda^2 S_1 \\ L^+ L^- S_2 &= -\lambda^2 S_2 \end{aligned} \tag{5}$$

where $L^\pm = \partial_\theta \mp \csc\theta + (1/2)\cot\theta$. One has $S_{1m} \cong S_{1-m}$. Moreover for $|m| \geq 1$, $\lambda^2 = (l + 1/2)^2$, $l = |m|, |m|+1, \ldots$ and for $m = 0$, $\lambda^2 = (l+1)^2$, $l = 0, 1, 2, \ldots$ the S_i's being essentially the Jacobi and the Tchebychef polynomials respectively [5]. The angular functions $S^{(i)}_{lm} = e^{im\varphi} S_{i,lm}(\theta)$ can then be assumed to be ortho-normalized

$$\int d\Omega \, S^{(i)}_{lm}(\theta,\varphi) \overline{\left(S^{(i)}_{l'm'}(\theta,\varphi)\right)} = \delta_{ll'}\delta_{mm'}, \quad i = 1, 2. \tag{6}$$

For what concerns the separated radial equations one is left with [10]

$$\begin{aligned} H_1' + \frac{i\sigma r}{r-2M} H_1 &= \left(i\mu_\star - \frac{\lambda}{r\sqrt{2}}\right) H_2 \\ H_2' + \frac{M - i\sigma r^2}{r(r-2M)} H_2 &= -\frac{2r}{r-2M}\left(i\mu_\star + \frac{\lambda}{r\sqrt{2}}\right) H_1 \end{aligned} \tag{7}$$

$(H' = dH/dr)$. The equations can be disentangled by sustitution to get

$$\begin{aligned} & H_1'' + \left[\frac{M}{r(r-2M)} - \frac{\lambda}{r(i\mu_\star\sqrt{2}r - \lambda)}\right] H_1' - \left[\frac{i\sigma(M + i\sigma r^2)}{(r-2M)^2} + \right. \\ & \qquad \left. + \frac{i\sigma\lambda}{(r-2M)(i\mu_\star\sqrt{2}r - \lambda)} + \frac{2\mu_\star^2 + \lambda^2}{r(r-2M)}\right] H_1 = 0 \\ & H_2'' + \left[\frac{3M}{r(r-2M)} - \frac{\lambda}{r(i\mu_\star\sqrt{2}r - \lambda)}\right] H_2' - \left[\frac{i\sigma r^2 + M}{r^2(r-2M)} - \frac{i\sigma(M - i\sigma r^2)}{(r-2m)^2} \right. \\ & \qquad \left. + \frac{\sqrt{2}i\mu_\star(M - i\sigma r^2)}{r(r-2M)(i\mu_\star\sqrt{2}r - \lambda)} + \frac{2\mu_\star^2 r^2 + \lambda^2}{r(r-2M)}\right] H_2 = 0 \end{aligned} \tag{8}$$

Both the form (7) and (8) of the radial equations do not fall into a class of equations whose solution (as far as the author knows) is known. A series integration has been proposed in Ref. [10]. However some local and global properties of the solutions can be put into evidence.

By taking into account that the eqs. (8) have a Fuchs-like behaviour for $r \sim 0$, $r \sim 2M$, one is led to represent the solutions H_1, H_2 in the form

$r^\alpha(r-2M)^\beta f(r)$ with $\alpha,\ \beta$ solutions of the corresponding indicial equations. A first possibility is

$$H_i \;=\; r^{\frac{1}{2}}(r-2M)^{-2i\sigma M}\, f_i(r), \quad i=1,2 \tag{9}$$

The f_i are determined, for $r\to 0$, by setting $H_i = A_i\, r^{\frac{1}{2}},\ i=1,2$. It must be $A_1=\lambda\sqrt{2}A_2$ in order to satisfy the system (7). Similarly for $r\to 2M,\ H_i = B_i(r-2M)^{-2i\sigma M}$ with $B_2(4i\sigma M-\frac{1}{2}) = (\sqrt{2}\lambda + 4i\mu_\star M)B_1$.

A second possibility is given by

$$\begin{aligned} H_1 \;&=\; r^{\frac{1}{2}}(r-2M)^{2i\sigma M+\frac{1}{2}}\, g_1(r) \\ H_2 \;&=\; r^{\frac{1}{2}}(r-2M)^{2i\sigma M-\frac{1}{2}}\, g_2(r) \end{aligned} \tag{10}$$

The difference with case (9) is only for $r\sim 2M$ where, by setting $H_1 = D_1(r-2M)^{2i\sigma M+\frac{1}{2}},\ H_2 = D_2(r-2M)^{2i\sigma M-\frac{1}{2}}$ it must be $D_1(4i\sigma M+\frac{1}{2}) = (i\mu_\star - \frac{\lambda}{2\sqrt{2}M})D_2$. In both cases (9), (10) the f_i's and the g_i's are determined, for large r, by the fact that

$$\begin{aligned} H_i \;\stackrel{r\to\infty}{\longrightarrow}\; & C_i\,\exp\left[\pm\, i\, r\,\sqrt{\sigma^2-m_e^2} \pm \frac{iM}{\sqrt{\sigma^2-m_e^2}}(2\sigma^2-m_e^2)\log r\right] \\ & C_2\mu_\star = (\sqrt{\sigma^2-m_e^2}+\sigma)C_1 \end{aligned} \tag{11}$$

as it can be obtained by developing in powers of $1/r$ the equation (8). By taking into account the position (4) one has that in case $\sigma > m_e$ the corresponding solution represents radially propagating wave for large r. By a transformation of the scheme , in the line of the work by Chandrasekhar [2], the propagation can be interpreted as the motion of a particle in a potential barrier. A discussion of this situation by standard approximation method is done in [6, 7]. In case σ is a complex number, the mentioned solution represents field that partially propagates and partially is absorbed by the media. In case $\sigma^2 < m_e^2$ the solution decay (or explodes) exponentially for large r. This suggests to look for possible (scalar) product of states under which the states are bound.

3 Bound states

It is well known that a scalar product of states is canonically induced by a conserved current. In case of Dirac field, the spinor $J^{AB'}(\phi,\psi) = P^A\bar{U}^{B'} + \bar{Q}^{B'}V^A$ with $\phi = (P,Q),\ \psi = (U,V)$ represents a conserved current, $\nabla_{AB'}J^{AB'} = 0$, provided ϕ, ψ are solutions of the equation (1) [2, 4]. The associated covariant product is then

$$(\phi,\psi) = \int |g|^{\frac{1}{2}}\sigma^{\alpha}_{AA'}J^{AA'}(\phi,\psi)\, n_\alpha\, d\Sigma \tag{12}$$

g the determinant of the metric tensor, Σ a Cauchy surface and n_α a future directed unit vector orthogonal to Σ. By choosing Σ to be $t = t_0$, $n^\alpha = \left(\sqrt{\frac{r}{r-2M}}, 0, 0, 0\right)$, and the fact that the σ-matrix assumes the form $\sigma^t_{AA'} = diag\{\frac{r}{r-2M}; \frac{1}{2}\}$ (in the tetrad here employed) one obtains

$$(\phi,\psi) =$$

$$\frac{1}{\sqrt{2}}\int d\Omega \int_0^\infty dr r^2 \left[\sqrt{\frac{r}{r-2M}}(P_1\bar{U}_1 + \bar{Q}_1V_1) + \sqrt{\frac{r-2M}{4r}}(P_0\bar{U}_0 + \bar{Q}_0V_0)\right] \tag{13}$$

By taking into account the factorization (3), the orthogonality relation (5) one has finally

$$(\phi,\phi)\ \cong \int_0^\infty dr\left(\sqrt{\frac{r}{r-2M}}H_1\bar{H}_1 + \frac{1}{2}\sqrt{\frac{r-2M}{r}}H_2\bar{H}_2\right) \tag{14}$$

If now the behaviour (9) of the solution is considered, one can check convergence of the integral (14) in $r = 0,\ r = 2M$. Similarly for the solution (10). Since in each case both H_1 and H_2 admit exponential decay for $\sigma^2 < m_e^2$, we have that bound states exist for spin 1/2 field equation in Schwarzschild spacetime. For $\sigma^2 > 0$, σ is real and the bound states have essentially, for large r, the form $\exp\left(i\sigma t - r\sqrt{m_e^2 - \sigma^2}\right)$ that is they represent locally oscillating fields that decay exponentially in r. For $\sigma^2 < 0,\ \sigma$ is an imaginary number and the bound solutions represent field that decay (or explode) exponentially in time and decay exponentially in r.

4 Remarks

The peculiarity of the bound states is of being labelled by an index that ranges over a continous set of values. This is in contrast with ordinary quantum mechanics where the bound states are generally associated to a discrete spectrum of an observable. In the present scheme the discrete spectrum appears if one requires the radial solutions H_1, H_2 to have a polynomial like behaviour for large r. Indeed, by developing in powers of $1/r$, both eqs. (8) are approximated by

$$H'' + \left[\sigma^2 - m_e^2 + \frac{2M}{r}(2\sigma^2 - m_e^2\right] H \;=\; 0 \tag{15}$$

whose solution is of the form $H = re^{\chi r} M\Big(1 + \frac{M}{\chi}(2\sigma^2 - m_e^2); 2; -2\chi r\Big),\ \chi = \pm i\sqrt{\sigma^2 - m_e^2},\ M(a;b;x)$ the Kummer function [1]. If one requires $1+\frac{M}{\chi}(2\sigma^2 - m_e^2) = -n,\ n = 1, 2, 3, ..$ then under the conditions $\sigma^2 < m_e^2,\ m_e M \ll 1$ one obtains the Hydrogen like energy spectrum

$$\sigma^2 \;\cong\; \sigma_n^2 \;=\; m_e^2\left(1 - \frac{m_e^2 M^2}{(n+1)^2}\right), \quad n = 1, 2, 3.. \tag{16}$$

The corresponding bound states are $H_n \sim r \exp\Big[- \frac{m_e^2 M}{n+1} r\Big] L_n^1\Big(\frac{2m_e^2 M}{n+1} r\Big),\ L_n^1$ the associated Laguerre polynomials. This is the counterpart of what happens for the spin 1 case [15, 3] and also for the spin 0 in Schwarzschild space-time.

Finally it can be remarked that the product of the radial functions does not coincide with what seems mathematically "natural". Indeed by simple manipulation of the eqs. (6) one would be led to consider the radial product

$$\int_0^\infty dr\left(\frac{r}{r - 2M}|H_1|^2 + \frac{1}{2}|H|_2^2\right) \tag{17}$$

that does not converge in $r = 2M$ in case of the solution (10). The difference between physical and mathematical product of states is already present at the level of the scalar field in Schwarzschild metric [12].

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