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On the Use of Hydrodynamic Instability Test as an Efficient Tool for Evaluating Viscoelastic Fluid Models

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Abstract

The so-called "second-order" rheological model has been of widespread use in studies related to viscoelastic fluids. The model, however, has been shown by Fosdick and Rajagopal (1979) to violate certain Thermodynamics constraints. The so-called "second-grade" model has been proposed as a remedy to comply with thermodynamics. But no experimental data has ever been presented to verify the appropriateness of the parameters used in this particular model. In the present work, it will be shown that instead of relying on hard-to-measure elastic properties of a fluid to see which model is the right one, it is much easier to resort to their instability response in plane Poiseuile flow for this purpose. To that end, it will be shown that in plane Poiseuille flow, the instability picture of second-grade fluids are dramatically different from second-order fluids. That is, while for second-order model, fluid's elasticity has a destabilizing effect, for second-grade model, in contrast, it is predicted to have a stabilizing effect.

Mathematics Subject Classification: 76E25

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1. Introduction

Hydrodynamic instability is a common phenomenon with fluid flows. That is, many flows of practical interest lose their stability whenever certain condition(s) are met. Pipe flows, for example, is known to become unstable when the Reynolds number exceeds certain values [Draad et al, 1998]. The end result of such flow instabilities at high Reynolds numbers (the so-called inertial instability) is often a switchover from a laminar flow to a turbulent one. This changeover in flow regimes is always accompanied by a dramatic alteration in flow characteristics such as pressure drop and/or the separation point thus making the phenomenon of significant industrial importance. This is perhaps why predicting conditions under which a flow may become unstable at high Reynolds numbers has been the focus of so much research in the past, and this is particularly so for Newtonian fluids [Drazin, 1981].

Like Newtonian fluids, non-Newtonian fluids, too, are vulnerable to inertial instability. Indeed, this kind of instability is of paramount importance in "turbulent drag reduction phenomenon" using polymeric additives [Sureshkumar et al, 1997]. In contrast to Newtonian fluids, however, non-Newtonian fluids are vulnerable to instability even at low Reynolds numbers [Rothenberger, 1973]. This kind of instability (the so-called "elastic instability") is frequently encountered in polymer processing operations [Agassant et al, 1991]. Obviously, one can also envisage cases in which both types of instability are in effect simultaneously [Groisman, 1998]. Interestingly, the role played by a fluid’s elasticity in its elasticity response is sometimes realized to be rather intriguing. For example, while in turbulent drag reduction phenomenon using certain high-molecular-weight polymers, a fluid’s elasticity is known to delay transition to turbulence for good, in extrusion of molten polymers through a die (say, for the production of plastic sheets) a fluid’s elasticity may give rise to the undesirable "sharkskin phenomenon" with deteriorating effects on surface quality [Barone et al, 1999]. It is perhaps because of such peculiarities that in recent years a growing interest is seen in the academic world on the subject of elastic instability [Shaqfeh, 1996].

Hydrodynamic instability is a difficult subject by all standards, and this is particularly so when dealing with viscoelastic fluids. That is, experimental data are rather rare and theoretical results are not always so convincing [Shaqfeh, 1996]. The main difficulty with viscoelastic fluids lies obviously in the diversity of these fluids in their constitutive behavior [Larson, 1988]. Further difficulty arises from their simultaneous viscous and elastic properties thus making it difficult to differentiate between those effects which are attributable to their shear-dependent viscosity from those which arise as a result of their elasticity. Under such conditions, the best that one can hope for is to make general conclusions as to the role played by a fluid’s elasticity on its breakdown from laminar flow to turbulent flow. And to achieve such a less ambitious goal, one is tempted to rely on simple viscoelastic fluid models such as the “second-order” model [Bird et al, 1987] with the advantage that it enables elastic effects to be addressed exclusively (i.e., without the complicating effects of the shear-dependent viscosity). Unfortunately, this model has been shown [Fosdick and Rajagopal, 1979] to violate certain thermodynamics constraints. The model was later revised by Dunn and Rajagopal [1995] in such a way that it could comply with thermodynamics. The revised model, which is increasingly referred to as the "second-grade" model, has never been tested experimentally nor has it been invoked in any instability study. As such, it is the main objective of the present work to show that the instability picture of second-grade fluids is at odds with those of second-order fluids, say, in plane Poiseuille flow.

2. Basic Equations

Our instability analysis starts from Cauchy equations of motion [Currie, 1993] which together with the continuity equation for an incompressible fluid constitute the four equations governing the isothermal flow of any fluid (whether Newtonian or non-Newtonian):

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right] = -\nabla p + \nabla.\tau \tag{1}$$

$$\nabla.\mathbf{v} = 0 \tag{2}$$

where ρ is the density, $\mathbf{V}$ is the velocity vector, and $\boldsymbol{\tau}$ is the extra stress tensor. As mentioned above, the fluids of interest will be assumed to obey the "second-order" and "second-grade" models as their constitutive equation. The extra stress tensor for such fluids is [Truesdell and Noll, 1992]:

$$\tau = \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2} \tag{3}$$

where μ is the (constant) viscosity of the fluid with α_1 and α_2 being the two normal stress moduli both reflecting a fluid’s elasticity. In this equation, $\mathbf{A_1}$ and $\mathbf{A_2}$ are kinematical tensors defined by [Rivlin and Ericksen, 1955]:

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^T \tag{4}$$

$$\mathbf{A_2} = \frac{D}{Dt}(\mathbf{A_1}) + \mathbf{A_1}.\nabla\mathbf{V} + (\nabla\mathbf{V})^T.\mathbf{A_1} \tag{5}$$

where D/Dt is the material derivative. The three material properties μ, α_1, and α_2 should normally be determined from viscometric data. Experimental data available for polymeric liquids suggest that $\alpha_1 < 0$, and $\alpha_2 \leq 0$. The model with such restrictions on its elastic parameters is commonly referred to as the "second-order" model [Bird et al, 1987]. Irrespective of the fact that it has been of widespread use in the past, the model has the shortcoming that it violates certain thermodynamics constraints [Fosdick and Rajagopal, 1979]. For the model to become compatible with thermodynamics, it has been shown by Dunn and Rajagopal [1995] that we should have $\alpha_1 > 0$ and $\alpha_1 + \alpha_2 = 0$. Fluids which meet these restrictions are increasingly referred to as "second-grade" fluids. As mentioned above, in the present work, we are going to contrast the instability picture of second-grade model with second-order model in plane Poiseuile flow. This particular flow has the advantage that it has been addressed by many researchers in the past [Thomas, 1953; Chun and Schwarz, 1968] meaning that comparison can be made with previous works. Plane Poiseuile flow has the further advantage that it renders itself to an exact analytical solution for both fluid models. That is, for pressure-driven flow between two parallel plates extending in the x-direction, it is easy to show that for these particular viscoelastic fluids the velocity profile is of the form $V(y) = 1 - y^2$ (made dimensionless using the centerline velocity). One can then proceed with imposing a small disturbance to the base flow and see what happens to this disturbance in the course of time [Orr, 1907; Sommerfeld, 1908]. As to the disturbance itself, we have decided to rely on two-dimensional disturbances both for its ease of analysis, and, more importantly, in order to compare our numerical results with published data in the literature. Thus we write:

$$
\begin{aligned}
u(x, y, t) &= V(y) + u'(x, y, t) \\
v(x, y, t) &= 0 + v'(x, y, t) \\
p(x, y, t) &= p_0(x) + p'(x, y, t)
\end{aligned}
\tag{6}
$$

These perturbed quantities are then inserted into the time-dependent equations of motion and terms nonlinear in the perturbed quantities are discarded. Having introduced a perturbation stream function ψ', it can be decomposed into its Fourier modes as: $\psi'(x, y, t) = f(y) \exp[i\alpha(x - ct)]$ where α is the (real) wave number, c is the (complex) wave velocity, and f is the (complex) amplitude of the perturbation. Substituting this stream function into the linearized set of momentum equations, the following fourth-order nonlinear ODE is obtained for our fluids of interest:

$$[1 - i\alpha K.\mathrm{Re}(V - c)](D^2 - \alpha^2)^2 f - i\alpha.\mathrm{Re}[(V - c)(D^2 - \alpha^2) - (V'' + K.V'''')] f = 0 \quad (7)$$

where the differential operator *D* has been used in place of *d/dy* for ease of reading. In this equation *Re* is the Reynolds number and *k* is the elasticity number defined respectively as: $\mathrm{Re} = \rho U_c h / \mu$ and $k = \alpha_1 / \rho h^2$.

It is to be noted that Eq. (7) is valid for both second-order and second-grade fluids by simply changing the sign of the elasticity number *k (*negative for second-order fluids and positive for second-grade fluids). Equation (7) is seen to be a fourth-order ODE, and so it needs four boundary conditions to be amenable to a numerical and/or analytical solution. The most relevant boundary conditions are the perturbation velocities being zero at the surface of both plates; that is: $f(\pm 1) = f'(\pm 1) = 0$. In the present work, use will be made of the symmetry of the velocity field thus it is sufficient to look for even modes only. That is, the boundary conditions which will be used in practice are: $f'(0) = f'''(0) = 0$ and $f(1) = f'(1) = 0$.

3. Method of solution

Equation (7) with its pertinent boundary conditions constitute a general eigenvalue problem of the form: $\mathrm{A.f} = \mathrm{cB.f}$ where *c* is the eigenvalue and f is the eigenfunction. The coefficients A and B appearing in this equation are related to *k, Re,* and α as:

$$\mathrm{A} = (-1 + \mathrm{i}\alpha\,\mathrm{K.Re.V})(\mathrm{D}^2 - 2\alpha^2\mathrm{D}^2 + \mathrm{I}) + \mathrm{i}\alpha.\mathrm{Re.V}(\mathrm{D}^2 - \alpha^2\mathrm{I}) + \mathrm{D}^2\mathrm{V} + \mathrm{K.D}^4\mathrm{V}$$
$$\mathrm{B} = \mathrm{i}\alpha\,\mathrm{K.Re}(\mathrm{D}^4 - 2\alpha^2\mathrm{D}^2 + \alpha^4\mathrm{I}) + \mathrm{i}\alpha.\mathrm{Re}(\mathrm{D}^2 - \alpha^2\mathrm{I}) \quad (8)$$

In these equations, D^2 and D^4 are differential operators which stand for d^2/dy^2 and d^4/dy^4 respectively. To test for the instability of any (parallel or nearly-parallel) base flow, V(y), one can fix *k* and *Re* and look for those wavenumbers α which make the imaginary part of the wavespeed *c* negative. In practice, however, it is sufficient to determine the neutral instability curve, i.e, the locus of those points in the $\alpha - \mathrm{Re} - k$ space for which the imaginary part of *c* is just zero. In the present work, two different numerical methods will be used for this purpose: *i)* the collocation method, and *ii)* the Ricatti method.

3.1 The Collocation Method

Spectral and pseudo-spectral methods are of widespread use in hydrodynamic instability studies because of their being of high accuracy [Canuto, 1988; Boyd, 2000]. The main idea behind spectral methods is to transform a problem in continuum domain into a discrete system of relatively small order. As to our eigenvalue problem as posed by Eq, (7), the complex eigenfunction, *f(y)*, is approxmiated by a sum of *N* base or trial functions ξ_n as: $f(y) \approx f_N(y) = \sum_{n=1}^{N} a_n \xi_n(y)$. And there are a variety of trial functions to choose from (e.g., Lagrange, Legender, Chebyshev, etc). In the present work, the trial functions ξ_n are constructed using Chebyshev polynomials in such a way that each of them will satisfy the required boundary conditions exactly; that is:

$$\xi_n(y) = T_{n-1}(y) - \frac{2(n+1)}{n+2} T_{n+1}(y) + \frac{n}{n+2} T_{n+3}(y) \tag{9}$$

where T_n is the Chebyshev polynomial of degree *n* defined in the interval [1,-1] by $T_n(y) = \cos[n \cos^{-1}(y)]$. By substituting the approximate solution, $f_N(y)$ in Eq. (7), a nonzero residue will be left reflecting the fact that the proposed solution is not indeed exact. The problem is then to choose the coefficients a_n such that this residue is minimized. In the standard fully spectral method, these coefficients can be obtained by requiring that the residue is orthogonal to each of the approximating functions φ_n. In the so-called pseudo-spectral method, the residue is forced to be become exactly equal to zero at certain points called collocation points. In the present work, we are going to rely on the latter method, i.e., the collocation-point method to find a_n. As to the selection of the collocation points, use will be made of the so-called Gauss-Lobatto quadrature points y_j defined by $y_j = \cos(j\pi/n); j = 0,1,2,...,N$. The end result of this routine is turned out to be a set of *N* linear algebraic equations in the form: $\sum_{n=1}^{N} b_{mn}(k,c,\alpha,Re)a_n = 0$ where the complex coefficients b_{mn} are in general functions of *k*, *Re*, α, and *c*. Since these *N* equations are linear and homogenous, a nontrivial solution exists if, and only if, the determinent of the $N \times N$ complex matrix b_{mn} vanishes identically. The search for eigenvalues, *c*, can then be initiated for any given combination of α and Re (for a fixed *k*) in order to determine the wave speed(s), c, for which the real and imaginary parts of this determinant vanish simultaneously. Having calculated all the eigenvalues of Eq. (7), the neutral instability curve can be plotted by identifying those points for which $Im(c) = c_i = 0$.

3.2 The Riccati's Method

The most popular method for computing the eigenvalues of the Orr-Sommerfeld equation is to rely on standard numerical methods such as the Runge-Kutta method while shooting with respect to c in order to satisfy the boundary conditions. Use of these methods removes any need to compute eigenvalues of large matrices and offers conceptual and analytical simplicity. To integrate the OS equation, the standard practice is to decompose it into a set of four quadratically nonlinear system of ordinary differential equation. To that end, the Orr-Sommerfeld equation is written as $\mathrm{d}\mathbf{F}/\mathrm{dt} = \mathbf{M}.\mathbf{F}$ where the vector **F** and the coefficient matrix ***M*** are defined as:

$$\mathbf{F} = (\mathrm{f}, \mathrm{f}', \mathrm{f}'' - \alpha^2 \mathrm{f}, \mathrm{f}''' - \alpha^2 \mathrm{f}') \tag{10}$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-\mathrm{i}\alpha\,\mathrm{Re}(\mathrm{V}'' + \mathrm{k}\mathrm{V}'''')}{1 - \mathrm{ik}\alpha(\mathrm{Re})(\mathrm{V} - \mathrm{c})} & 0 & \alpha^2 + \dfrac{\mathrm{i}\alpha\,\mathrm{Re}(\mathrm{V} - \mathrm{c})}{1 - \mathrm{i}\alpha\mathrm{k}\,\mathrm{Re}(\mathrm{V} - \mathrm{c})} & 0 \end{bmatrix} \tag{11}$$

We now assume that f_1 and f_2 are two independent solutions of Eq. (7) satisfying the initial conditions: $f_1(0) = [1, 0, 0, 0]^T$, $f_2(0) = [0, 0, 1, 0]^T$ both corresponding to the boundary conditions at y = 0. The general solution of the Orr-Sommerfeld equation must then be of the form: $f(y) = \lambda_1 f_1 + \lambda_2 f_2$. Now, having imposed the boundary conditions at y = 1, a homogenous system of equations will be obtained as:

$$\lambda_1 f_1(1) + \lambda_2 f_2(1) = 0 \;\; ; \;\; \lambda_1 f_1'(1) + \lambda_2 f_2'(1) = 0 \tag{12}$$

In a search for non-zero solutions, an eigenvalue problem is then obtained as:

$$\varphi(\alpha, c, k, \mathrm{Re}) = f_1(1) f_2'(1) + f_1'(1) f_2(1) = 0 \tag{13}$$

For fixed values of α, k, Re, the eigenvalues c can be found by searching iteratively for the zeros of ϕ. Unfortunately, numerical solution of Eq. (13) is accompanied with some difficulties due to the fact that f_1 and f_2 loose their linear independency near y = 1. There are different methods to overcome this problem among which the Riccati method is among of the most successful ones. The basic idea in the Riccati method is to transform a linear eigenvalue problem into a set of nonlinear initial-value equations. To illustrate the method, lets assume that we have a system of equations of the form [Scott, 1973]:

$$\mathbf{P}' = A_1\mathbf{P} + A_2\mathbf{Q}\,; \quad -\mathbf{Q}' = A_3\mathbf{P} + A_4\mathbf{Q} \tag{14}$$

where **P** and **Q** are N-vectors, and A_1, A_2, A_3, A_4 are $N \times N$ coefficient matrices. The Riccati matrix is then introduced through the transformation $\mathbf{P} = \mathbf{R}.\mathbf{Q}$ with **R** satisfying the following first-order nonlinear matrix differential equation $\mathbf{R}' = A_2 + A_1\mathbf{R} + \mathbf{R}A_4 + \mathbf{R}A_3\mathbf{R}$ where we have: $\mathbf{P}(0) = 0$, $\mathbf{R}(0) = 0$, and $\mathbf{Q}(0) = \mathbf{I}$ with $\boldsymbol{I}$ being the identity matrix.. The eigenvalues, c, can then be determined by requiring that $\det[\mathbf{R}(1)] = 0$. To apply this method to the modified Orr-Sommerfeld equation, we should substitute:

$$\mathbf{P} = \begin{bmatrix} f' \\ f'' - \alpha^2 f' \end{bmatrix};\ \mathbf{Q} = \begin{bmatrix} f \\ f'' - \alpha^2 f \end{bmatrix} \tag{15}$$

The modified OS equation is then recovered by setting:

$$A_1 = 0\,;\ A_3 = I\,;\ A_4 = 0 \tag{16}$$

$$A_2 = \begin{bmatrix} \alpha^2 & 1 \\ \dfrac{-i\alpha\,\mathrm{Re}(V'' + kV'''')}{1 - ik\alpha\,\mathrm{Re}(V - c)} & \alpha^2 + \dfrac{i\alpha\,\mathrm{Re}(V - c)}{1 - ik\alpha\,\mathrm{Re}(V - c)} \end{bmatrix} \tag{17}$$

where $\mathbf{R}' + \mathbf{R}^2 = A_2$. Now, defining $\mathbf{R} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$, a set of four nonlinear coupled first-order ordinary differential equations can be obtained as:

$$\begin{aligned}
&r_1' + r_1^2 + r_2 r_3 = \alpha^2 \\
&r_2' + r_1 r_2 + r_2 r_4 = 1 \\
&r_3' + r_1 r_3 + r_3 r_4 = -\frac{i\alpha\,\mathrm{Re}(V'' + kV'''')}{1 - i\alpha k\,\mathrm{Re}(V - c)} \\
&r_4' + r_2 r_3 + r_4^2 = \alpha^2 + \frac{i\alpha\,\mathrm{Re}(V - c)}{1 - i\alpha k\,\mathrm{Re}(V - c)}
\end{aligned} \tag{18}$$

The boundary conditions required to solve these equations are:

$$@\, y = 0\,, \qquad r_1 = r_2 = r_3 = r_4 = 0$$

$$@\, y = 1, \qquad r_2 = 0 \tag{19}$$

The system of Eqs. (18) subject to the boundary conditions as given by Eq. (19) can easily be solved numerically using Mathematica.

4. Results and discussion

The two methods of solution described above were translated into two separate Fortran codes and run on a Pentium-4 PC. For the first method, i.e., the collocation-point method, we used 100 Chebyshev terms to increase accuracy of the results. To see if the computer codes developed in this work are functioning properly, they were used to determine the critical Reynolds number for Newtonian fluids (i.e., for k = 0). This was done at Re = 10,000 for a wavelength of $\alpha = 1$. As is well established in the literature, the critical Reynolds number for Poisuille flow of Newtonian fluids is known to be equal to 5772.20 [Orszag, 1971]. Our collocation-point method renders a critical Reynolds number of 5772.84. The Riccati method, gives a critical Reynolds number of 5772.40. Obviously both methods are working properly, at least, as far as Newtonian fluids are concerned. To further validate the codes, they were used to obtain neutral instability curves for second order fluids too. Figure 1 presents typical results for second-order fluids. These results are virtually the same as those published in the literature obtained using finite differences [Chun and Schwarz, 1968]. Having validated the code, it was used to find neutral instability curves for the fluid of interest, i.e., the second-grade fluid. Table 1 presents a comparison between critical Reynolds and wave numbers between second-order and second-grade fluids. As seen in this table, while for second-order fluids, elasticity has a destabilizing effect, for second-grade fluids, it has a stabilizing effect.

To elucidate the main cause of instability, Fig. 3 shows a plot of the near-wall perturbation vorticity for a wave number of α = 1 at Re = 5000 for a second-order fluid at three different elasticity numbers of $k = 0, 10^{-5}, 5\times10^{-4}$. Under these conditions, the flow is known to be stable for k = 0 and $k = 10^{-5}$ (see Table 1). As seen in Fig. 3, for these two stable cases, perturbation vorticity is decaying in time. In contrast, for the case of $k = 10^{-4}$ for which the flow is known to be unstable (see Table 1), the vorticity is seen to be growing in time. Therefore, it appears that instability has roots in the time evolution of the near-wall vorticity of the perturbed flow.

Table 1: Instability behavior of second-order (subscript SO) and econd-grade (subscript SG) fluids in plane Poiseuille flow.

α_{SG}	$(Re)_{SG}$	α_{SO}	$(Re)_{SO}$	$\lvert k \rvert$
1.0195	5772	1.0195	5772	0
1.0155	5915	1.0240	5639	10^{-5}
0.9893	6587	1.0475	5168	5×10^{-5}
0.9510	7756	1.0750	4698	10^{-4}
0.8415	13392	1.1215	4014	2×10^{-4}

5. Concluding remarks

In the present work, hydrodynamic instability of second-grade fluids has been investigated in plane Posieuile flow, to the best of our knowledge, for the first time. Two different numerical methods have been used for this purpose, namely: *i)* the collocation method, and *ii)* the Riccati's method—the latter, to the best of our knowledge, for the first time for viscoelastic fluids. Consistent results were obtained using both methods for the critical Reynolds number and also the wave number. Based on the results presented in this work, it is concluded that for second-order fluids, fluid's elasticity has a destabilizing effect. In contrast, for second-grade fluids, fluid's elasticity is predicted to have a stabilizing effect. Since the response of these two controversial fluid models to infinitesimal disturbances are at odds with each other in plane Poiseuille flow, it is proposed that instead of measuring elastic properties of a fluid for determining which model is the right one (based on their sign being positive or negative), one may equally resort to their different instability response in plane Poiseuile flow as an effective and simpler means for this purpose.

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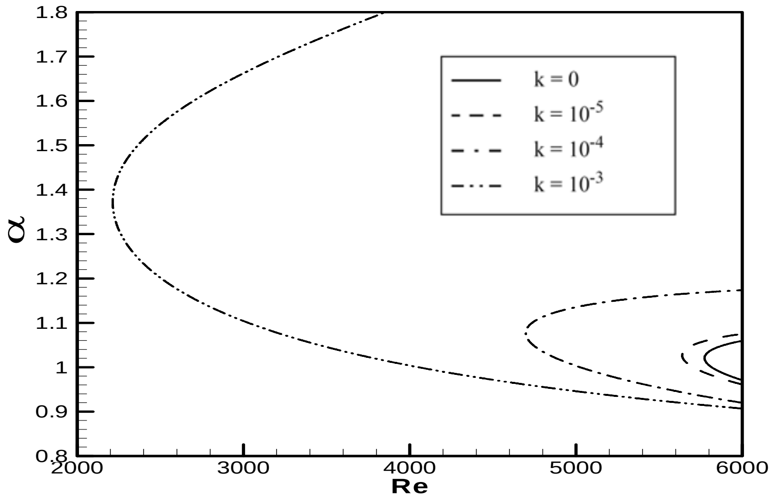

Fig. 1: Neutral instability curves for second-order fluids as a function of the elasticity number k obtained using the collocation method.

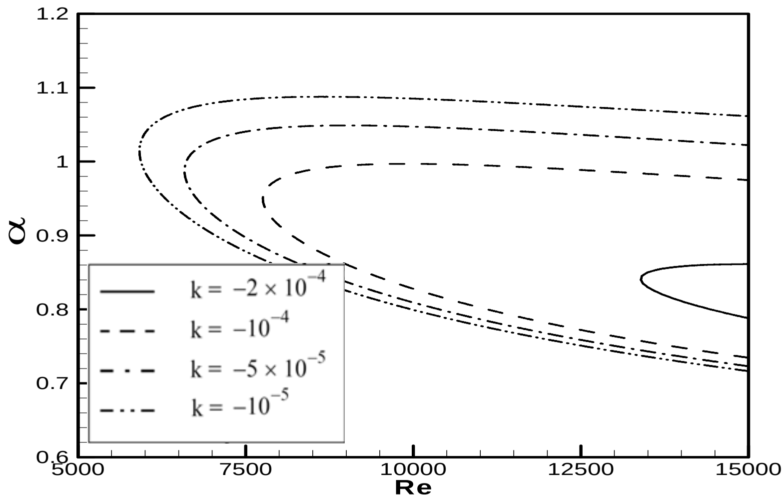

Fig. 2: Neutral instability curves for second-grade fluids as a function of the elasticity number k obtained using the Riccati's method.

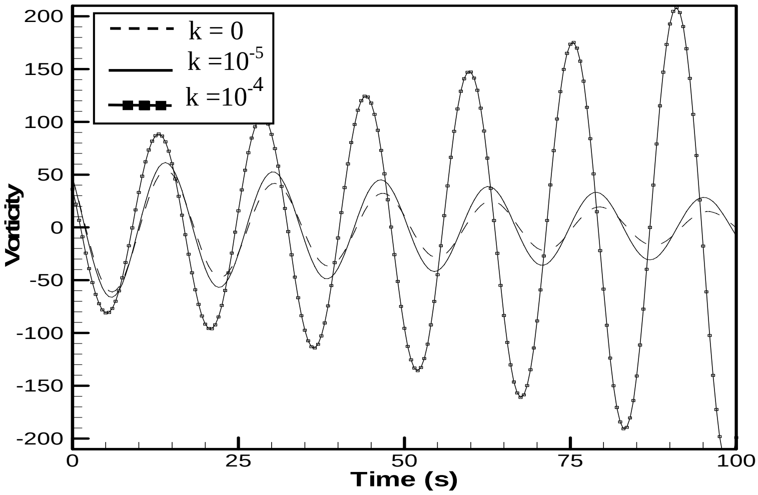

Fig. 3: Time evolution of perturbation vorticity as a function of the elasticity number.

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