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General Solution of the Stress Potential Function in Lekhnitskii's Elastic Theory for Anisotropic and Piezoelectric Materials

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Abstract

It is found that the original Lekhnitskii general solution (Lekhnitskii, 1963) is far from being completeness, because some important terms were lost. New types of complete general solution for anisotropic as well as piezoelectric materials are then developed analytically. The new general solutions include the origin one completely. Unlike the original one, the new general solution can be proved to degenerate normally to the Muskhelishvili theory for isotropic materials. Hence the Lekhnitskii and the Muskhelishvili theory are unified, and the well-known imperfect in the original Lekhnitskii theory which perplexed researchers for a long time is avoided.

Keywords: Lekhnitskii's theory; General solution; Piezoelectric materials; Anisotropic materials; Completeness

1. Introduction

It is well known that both the elastic mechanics and the fracture mechanics of anisotropic as well as piezoelectric materials are very important in theory analyses and practical engineering applications in devising some structures and components. As a foundation stone of the elastic mechanics and the fracture mechanics, Lekhnitskii's theory (Lekhnitskii, 1963) has been widely used in dealing with various kinds of problems in the anisotropic and piezoelectric elastic mechanics over the past several decades. Based on the general solution of the stress potential function in the theory, a lot of research work has been done.

However, as pointed out by some previous researchers (Ting, 1996; Yin, 2000), Lekhnitskii's theory has a serious questionable point, i.e., it can not degenerate back to the Muskhelishvili theory (Muskhelishvili, 1953) for isotropic elastic mechanics. Let us see it in more details and see what will happen. In Muskhelishvili's theory, the Airy stress function $U_M(x, y)$ for a plane problem is expressed in the form (well-known Goursat formulism)

$$U_{\mathrm{M}}(x,y)=\mathrm{Re}[\bar{z}\phi(z)+\chi(z)], \tag{1}$$

where "Re" represents the real part of the related complex function and the overbar means the transpose; $\phi(z)$ and $\chi(z)$ are arbitrary analytical functions, both of which are defined on the whole z-plane, and $z = x + iy$ is a complex variable. In Lekhnitskii's theory for a plane strain problem in anisotropic materials, the Airy stress function $U_L(x, y)$ is expressed as

$$U_{\mathrm{L}}(x,y)=2\,\mathrm{Re}[U_1(z_1)+U_2(z_2)], \tag{2}$$

where $U_1(z_1)$ and $U_2(z_2)$ are arbitrary analytical functions defined on the z_1- and z_2-plane, respectively. $z_1= x +\mu_1 y$ and $z_2 = x + \mu_2 y$ are two complex variables and μ_1, μ_2 with their conjugates are the four characteristic roots of the related eigenvalue problem. Comparing Eqs.(1) and (2) shows clearly that, for isotropic materials and thus $\mu_1 = \mu_2$, Eq.(2) can not be induced directly into Eq.(1), and which, for a long time, has been seeing as a imperfect in Lekhnitskii's theory. Ting (1996) and Yin (2000) discussed this problem in scarcely way and presented some solutions for so-called degenerate and extra-degenerate materials including isotropic materials. However, up to now, although much of such work is done, the correctness, or in other words, the completeness of the Lekhnitskii theory is not verified and some questions are still existed: why the Lekhnitskii theory can not be degenerated into the

Muskhelishvili theory, which is an obvious fact in view of the physical point? Why the expression for the Airy stress function in anisotropic materials, which is relatively more complicated, is simpler than that in isotropic materials and is it really available? Eq.(1) shows that the Airy stress function in isotropic materials is not an analytical function. It is so strange that the Airy stress function for an anisotropic material, which is much more complicate than the isotropic one, can go so far as to be an analytical function, why? One of reasonable replies to above questions should be clearly that some mistakes must be existed in the original Lekhnitskii theory, which is just an important problem solved in this paper. Moreover, recently, more and more contradictions between experimental data and theoretical predication in the piezoelectric fracture mechanics are observed, which has perplexed researchers for a long time, and it also be a promotion to re-examine the theoretical tools adopted in those analyses in more details, although some other researchers maybe interesting in contributing these contradictions to some microscope factors such as electric domain switch or the nonlinearity of crack-tip field.

In the following sections, a detailed theoretical manipulation is developed and a new type of the general solution of the stress potential function in Lekhnitskii's theory for anisotropic materials is obtained. Some short discussions are then followed. Additionally, the general potential function solution under generalize plane strain in a piezoelectric material is also presented.

2. Basic formulism

The Manipulation about the new general potential function solution is so tedious that, for the briefness, only the simplest case, as an example, is presented here. Considering an *x*-*y* plane elastic problem under plane strain state for an anisotropic material, from Lekhnitskii's formula (Lekhnitskii, 1963), the general solution of the stress potential function will be derived by solving directly the differential equation as follow:

$$D_4 D_3 D_2 D_1 U = 0\,, \tag{3}$$

where the differential operator $D_j = \partial/\partial y - \mu_j \partial/\partial x$ (j =1, 2, 3, 4) and μ_j are the characteristic roots in Lekhnitskii's theory; $U(x, y)$ is Airy's stress function, which is defined by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \tag{4}$$

and σ_{xx}, σ_{yy}, σ_{xy} are corresponding components of the stress tensor. Eq.(3) is equivalent to the following group of differential equations

$$D_1 U = \phi_2, \quad D_2\phi_2 = \phi_3, \quad D_3\phi_3 = \phi_4, \quad D_4\phi_4 = 0. \tag{5}$$

In order to develop the solution of Eq.(5), generally, by using complex variables $z_j = x + \mu_j y$ (j = 1, 2, 3, 4), one can transform differential operators D_j into the following form:

$$D_j = (\bar{\mu}_j - \mu_j)\frac{\partial}{\partial \bar{z}_j}, \quad \bar{D}_j = (\mu_j - \bar{\mu}_j)\frac{\partial}{\partial z_j}, \tag{6}$$

and it can be easily shown that

$$z_j = A_{jk} z_k - B_{jk}\bar{z}_k, \quad dz_j = A_{jk} dz_k - B_{jk} d\bar{z}_k \qquad (j,k = 1,2,3,4;\ j \neq k), \tag{7a}$$

where

$$A_{jk} = \frac{\mu_j - \bar{\mu}_k}{\mu_k - \bar{\mu}_k}, \quad B_{jk} = \frac{\mu_j - \mu_k}{\mu_k - \bar{\mu}_k}. \tag{7b}$$

Equation (5) shows that any complex variable z_j can be represented as a function of other two independent variables z_k and $\bar{z}_k$, which implies that an arbitrary complex function of variable z_j and $\bar{z}_j$ can be transferred directly into the function of the variables z_k and $\bar{z}_k$, and this will be very important in the following manipulations. Lekhnitskii (1963) showed that the characteristic roots μ_j (j = 1, 2, 3, 4) must be complex numbers so that they can be expressed as $\mu_1, \mu_2, \mu_3 = \bar{\mu}_1$ and $\mu_4 = \bar{\mu}_2$. Thus, Eqs.(3) and (5) can be rewritten, respectively, in the form

$$\bar{D}_2 \bar{D}_1 D_2 D_1 U = 0, \tag{8}$$

and

$$D_1 U = \phi_2, \quad D_2\phi_2 = \phi_3, \quad \bar{D}_1\phi_3 = \phi_4, \quad \bar{D}_2\phi_4 = 0. \tag{9}$$

Let us now solve Eq.(9) step by step.

Firstly, in virtue of Eq.(6), the last equation in Eq.(9) gives out

$$C_2 \frac{\partial \phi_4}{\partial z_2} = 0, \tag{10}$$

where $C_j = \mu_j - \bar{\mu}_j$ (j = 1, 2) are complex constants, which are determined completely by material constants. One hence reaches that (remember that, as described above, ϕ_4 can be seen as a function of the variables z_2 and $\bar{z}_2$ now)

$$\phi_4 = f_4(\bar{z}_2), \tag{11}$$

where $f_4(\bar{z}_2)$ is an arbitrary function.

Secondly, by substituting Eq.(11) into the third equation in Eq.(9), one has

$$C_1 \frac{\partial \phi_3}{\partial z_1} = \phi_4 = f_4(\bar{z}_2). \tag{12}$$

Thus, with the second equation in Eq.(7a), the function ϕ_3 can be integrated as:

$$\begin{aligned}\phi_3 &= \frac{1}{C_1}\int f_4(\bar{z}_2) dz_1 + f_3(\bar{z}_1) \\ &= \frac{1}{C_1}\int f_4(\bar{z}_2)(A_{12} dz_2 - B_{12} d\bar{z}_2) + f_3(\bar{z}_1), \\ &= \frac{A_{12}}{C_1} z_2 f_4(\bar{z}_2) - \frac{B_{12}}{C_1} g_4(\bar{z}_2) + f_3(\bar{z}_1)\end{aligned} \tag{13}$$

where another arbitrary function $f_3(\bar{z}_1)$ is introduced, and we denote that

$$g_4(\bar{z}_2) = \int f_4(\bar{z}_2) d\bar{z}_2 . \tag{14}$$

Thirdly, from the second equation in Eq.(9) and the expression of the function ϕ_3 as described in Eq.(13), one arrives to

$$\bar{C}_2 \frac{\partial \phi_2}{\partial \bar{z}_2} = \phi_3 = \frac{A_{12}}{C_1} z_2 f_4(\bar{z}_2) - \frac{B_{12}}{C_1} g_4(\bar{z}_2) + f_3(\bar{z}_1). \tag{15}$$

Similarly, this gives the function ϕ_2 as following:

$$\phi_2 = \frac{A_{12}}{\bar{C}_2 C_1} z_2 \int f_4(\bar{z}_2) d\bar{z}_2 - \frac{B_{12}}{\bar{C}_2 C_1} \int g_4(\bar{z}_2) d\bar{z}_2 + \frac{1}{\bar{C}_2} \int f_3(\bar{z}_1)(\bar{A}_{21} d\bar{z}_1 - \bar{B}_{21} dz_1) + f_2(z_2) \qquad ,(16)$$

$$= \frac{A_{12}}{\bar{C}_2 C_1} z_2 g_4(\bar{z}_2) - \frac{B_{12}}{\bar{C}_2 C_1} h_4(\bar{z}_2) + \frac{\bar{A}_{21}}{\bar{C}_2} g_3(\bar{z}_1) - \frac{\bar{B}_{21}}{\bar{C}_2} z_1 f_3(\bar{z}_1) + f_2(z_2)$$

and some other arbitrary functions f_2, g_3 and h_4 are introduced and denoted as

$$g_3(\bar{z}_1) = \int f_3(\bar{z}_1) d\bar{z}_1, \quad h_4(\bar{z}_2) = \int g_4(\bar{z}_2) d\bar{z}_2 \ . \qquad (17)$$

At last, by substituting Eq.(17) into the first equation in Eq.(9), one has

$$\bar{C}_1 \frac{\partial U}{\partial \bar{z}_1} = \frac{A_{12}}{\bar{C}_2 C_1} z_2 g_4(\bar{z}_2) - \frac{B_{12}}{\bar{C}_2 C_1} h_4(\bar{z}_2) + \frac{\bar{A}_{21}}{\bar{C}_2} g_3(\bar{z}_1) - \frac{\bar{B}_{21}}{\bar{C}_2} z_1 f_3(\bar{z}_1) + f_2(z_2), \quad (18)$$

and then, the general potential function solution U can be obtained by solving above equation, as

$$\begin{aligned} U_{\bar{2}\bar{1}21} = & -\frac{A_{12}\bar{B}_{12}}{\bar{C}_1\bar{C}_2 C_1}\frac{z_2^2}{2} g_4(\bar{z}_2) + \frac{A_{12}\bar{A}_{12} + B_{12}\bar{B}_{12}}{\bar{C}_1\bar{C}_2 C_1} z_2 h_4(\bar{z}_2) - \frac{B_{12}\bar{A}_{12}}{\bar{C}_1\bar{C}_2 C_1} k_4(\bar{z}_2) \\ & + \frac{\bar{A}_{21}}{\bar{C}_1\bar{C}_2} h_3(\bar{z}_1) - \frac{\bar{B}_{21}}{\bar{C}_1\bar{C}_2} z_1 g_3(\bar{z}_1) + \frac{\bar{A}_{12}}{\bar{C}_1} \bar{z}_2 f_2(z_2) - \frac{\bar{B}_{12}}{\bar{C}_1} g_2(z_2) + f_{11}(z_1) \end{aligned} \quad , \quad (19)$$

where the newly introduced arbitrary function are expressed as:

$$h_3(\bar{z}_1) = \int g_3(\bar{z}_1) d\bar{z}_1, \quad k_4(\bar{z}_2) = \int h_4(\bar{z}_2) d\bar{z}_2, \quad g_2(z_2) = \int f_2(z_2) dz_2 \ , \quad (20)$$

Moreover, from the symmetrical nature of the characteristic roots, Eq.(9) can be written, for example, as

$$\bar{D}_2 \bar{D}_1 D_1 D_2 U = 0, \qquad (21a)$$

and then the corresponding equation group is

$$D_2 U = \phi_2, \quad D_1 \phi_2 = \phi_3, \quad \bar{D}_1 \phi_3 = \phi_4, \quad \bar{D}_2 \phi_4 = 0. \qquad (21b)$$

Thus, for this equation group, one can get its solution in the same manner as described above, as:

$$U_{\bar{2}\bar{1}12} = \frac{A_{12}\bar{A}_{12} + B_{12}\bar{B}_{12}}{\bar{C}_2\bar{C}_1 C_1} z_2 h_4(\bar{z}_2) - \frac{A_{12}\bar{B}_{12}}{\bar{C}_2\bar{C}_1 C_1}\frac{z_2^2}{2} g_4(\bar{z}_2) - \frac{B_{12}\bar{A}_{12}}{\bar{C}_2\bar{C}_1 C_1} k_4(\bar{z}_2) + \frac{\bar{A}_{21}}{\bar{C}_2\bar{C}_1} h_3(\bar{z}_1) - \frac{\bar{B}_{21}}{\bar{C}_2\bar{C}_1} z_1 g_3(\bar{z}_1) + \frac{\bar{A}_{21}}{\bar{C}_2}\bar{z}_1 f_1(z_1) - \frac{\bar{B}_{21}}{\bar{C}_2} g_1(z_1) + f_{12}(z_2) \quad . \quad (22)$$

Here and following, all appeared new functions are arbitrary. Similarly, other symmetrical solution of the potential function U can be derived in the following forms:

$$U_{\bar{1}\bar{2}21} = \frac{A_{21}\bar{A}_{21} + B_{21}\bar{B}_{21}}{\bar{C}_1\bar{C}_2 C_2} z_1 h_5(\bar{z}_1) - \frac{A_{21}\bar{B}_{21}}{\bar{C}_1\bar{C}_2 C_2}\frac{z_1^2}{2} g_5(\bar{z}_1) - \frac{B_{21}\bar{A}_{21}}{\bar{C}_1\bar{C}_2 C_2} k_5(\bar{z}_1) + \frac{\bar{A}_{12}}{\bar{C}_1\bar{C}_2} h_6(\bar{z}_2) + \frac{\bar{B}_{12}}{\bar{C}_1\bar{C}_2} z_2 g_6(\bar{z}_2) + \frac{\bar{A}_{12}}{\bar{C}_1}\bar{z}_2 f_7(z_2) + \frac{\bar{B}_{12}}{\bar{C}_1} g_7(z_2) + f_8(z_1) \quad ; \quad (23a)$$

$$U_{\bar{1}\bar{2}12} = \frac{A_{21}\bar{A}_{21} + B_{21}\bar{B}_{21}}{\bar{C}_2\bar{C}_1 C_2} z_1 h_5(\bar{z}_1) - \frac{A_{21}\bar{B}_{21}}{\bar{C}_2\bar{C}_1 C_2}\frac{z_1^2}{2} g_5(\bar{z}_1) - \frac{B_{21}\bar{A}_{21}}{\bar{C}_2\bar{C}_1 C_2} k_5(\bar{z}_1) + \frac{\bar{A}_{12}}{\bar{C}_2\bar{C}_1} h_6(\bar{z}_2) - \frac{\bar{B}_{12}}{\bar{C}_2\bar{C}_1} z_2 g_6(\bar{z}_2) + \frac{\bar{A}_{21}}{\bar{C}_2}\bar{z}_1 f_9(z_1) - \frac{\bar{B}_{21}}{\bar{C}_2} g_9(z_1) + f_{10}(z_2) \quad ; \quad (23b)$$

By taking the conjugation of above four solutions (19), (22) and (23) and summing all of them up, one can derive the needed general potential function solution, as

$$U = \frac{1}{4}\mathrm{Re}\left\{\bar{z}_1^2\chi_1(z_1) + \bar{z}_1\chi_2(z_1) + \chi_3(z_1) + \bar{z}_2^2\psi_1(z_2) + \bar{z}_2\psi_2(z_2) + \psi_3(z_2)\right\}, \quad (24)$$

where

$$\chi_1(z_1) = -\frac{\bar{A}_{21}B_{21}}{C_1 C_2\bar{C}_2}\bar{g}_5(z_1), \quad (25a)$$

$$\chi_2(z_1) = 2\frac{\bar{A}_{21}A_{21} + \bar{B}_{21}B_{21}}{C_1 C_2\bar{C}_2}\bar{h}_5(z_1) + \frac{\bar{A}_{21}}{\bar{C}_2} f_9(z_1) - \frac{2B_{21}}{C_1 C_2}\bar{g}_3(z_1) + \frac{\bar{A}_{21}}{\bar{C}_2} f_1(z_1) \quad , \quad (25b)$$

$$\chi_3(z_1) = -\frac{2\bar{B}_{21}A_{21}}{C_1 C_2\bar{C}_2}\bar{k}_5(z_1) - \frac{\bar{B}_{21}}{\bar{C}_2} g_9(z_1) + f_8(z_1) + \frac{2A_{21}}{C_1 C_2}\bar{h}_3(z_1) - \frac{\bar{B}_{21}}{\bar{C}_2} g_1(z_1) + f_{11}(z_1) \quad , \quad (25c)$$

$$\psi_1(z_2) = \frac{\bar{A}_{12} B_{12}}{C_1 C_2 \bar{C}_1} \bar{g}_4(z_2), \tag{25d}$$

$$\begin{aligned}\psi_2(z_2) = 2\frac{\bar{A}_{12} A_{12} + \bar{B}_{12} B_{12}}{C_1 C_2 \bar{C}_1} \bar{h}_4(z_2) \\ + \frac{\bar{A}_{12}}{\bar{C}_1} f_2(z_2) - \frac{2B_{12}}{C_1 C_2} \bar{g}_6(z_2) + \frac{\bar{A}_{12}}{\bar{C}_1} f_7(z_2)\end{aligned}, \tag{25e}$$

$$\begin{aligned}\psi_3(z_2) = \frac{2\bar{B}_{12} A_{12}}{C_1 C_2 \bar{C}_1} \bar{k}_4(z_2) - \frac{2\bar{B}_{12}}{\bar{C}_1} g_2(z_2) \\ + f_{12}(z_2) + \frac{2A_{12}}{C_1 C_2} \bar{h}_6(z_2) - \frac{\bar{B}_{12}}{\bar{C}_1} g_7(z_2) + f_{10}(z_2)\end{aligned}. \tag{25f}$$

3. Discussion

A new type of the general solution of the stress potential function in Lekhnitskii's elastic theory of a plane strain problem in anisotropic materials is obtained analytically as shown in Eq.(24). It is seen that the new general solution not only includes completely the original one (when the terms concern with functions χ_1, χ_2, ψ_1 and ψ_2 are neglected) derived by Lekhnitskii (1963) himself, but also be much different from the original one. This means that the original general solution is far from being completeness with some important terms neglected. Moreover, it should be noticed that, for isotropic materials, $\mu_j = \mu_k$ (j, k = 1, 2 and $j \neq k$) and then z_2 = z_1, B_{jk} = 0 from Eq.(7b), and then $\chi_1(z_1) = \psi_1(z_1) \equiv 0$ from Eqs. (25a) and (25d), which implies that Eq.(24) induces directly into Eq.(1), i.e. the new complete Lekhnitskii solution can be degenerated to Muskhelishvili solution. The Muskhelishvili theory is thus included in the new complete Lekhnitskii theory and the two theories are unified now. This also provides an indirect verification of the validity of the present solution.

It can also be seen clearly, from the manipulating process of the new solution described above, that the form of the general stress potential function solution is depending on the practical problem considered. For instance, in treating a generalized plane problem in an anisotropic material in which the control differential equation is in six order, and its general potential function solution is of the form

$$U = \mathrm{Re}\sum_{j=1}^{3}\left\{\bar{z}_j^3\psi_3^{(j)}(z_j) + \bar{z}_j^2\psi_2^{(j)}(z_j) + \bar{z}_j\psi_1^{(j)}(z_j) + \psi_0^{(j)}(z_j)\right\}. \quad (26)$$

Namely, in a general potential function solution, the highest order of the variable $\bar{z}_j$ is equal to half the order of the control differential equation. Hence, for a generalized plane problem in a piezoelectric material, the generalized solution will take the form of

$$U = \mathrm{Re}\sum_{j=1}^{4}\left\{\bar{z}_j^4\psi_4^{(j)}(z_j) + \bar{z}_j^3\psi_3^{(j)}(z_j) + \bar{z}_j^2\psi_2^{(j)}(z_j) + \bar{z}_j\psi_1^{(j)}(z_j) + \psi_0^{(j)}(z_j)\right\}. \quad (27)$$

From the new form of a general potential function solution expressed by Eqs.(26) and (27) for corresponding materials under certain conditions, it can be observed that the newly derived solutions are much more general as well as complicate comparing with the original one. Under the new solution, accordingly, the expressions of some related physical quantities such as stress, strain and displacement as well as electrical quantities such as electric field and electric displacement etc must change in comparison with the existed one. Hence it can be expected that, under the new general potential function solution, some new theoretical results in anisotropic and piezoelectric fracture mechanics must be arrived, and which may rewritte the existed theoretical conclusions greatly. On the other hand, the well known Stroh theory (Stroh, 1958, 1962), which has also been used widely in anisotropic and piezoelectric elastic and fracture mechanics, should be reconsidered too in details, since its general solution was reached by adopting a inverse method either, in which some terms in the solutions maybe ignored too. Finally, it should be pointed out that whether the new solution is a complete one in mathematics is not sure certainly. Only can be shown is that the more completed general potential function solution comparing with the existed one is reached in this paper.

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