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A new turbulence model for Large Eddy Simulation

Francesco Gallerano

Dipartimento di Idraulica Trasporti e Strade
Universita degli Studi di Roma "La Sapienza"
Via Eudossiana 18, Roma
Italy
Francesco.Gallerano@uniroma1.it

Giovanni Cannata

Dipartimento di Idraulica Trasporti e Strade
Universita degli Studi di Roma "La Sapienza"
Via Eudossiana 18, Roma
Italy

Abstract

The present - day Large Eddy Simulation models based on the Smagorinsky assumption and the drawbacks of the dynamic calculation of the closure coefficient for the generalised subgrid scale turbulent stress tensor are presented. The relations between numerical scheme conservation property of mass, momentum and kinetic energy and the drawbacks of the dynamic Smagorinsky - type turbulence models are shown. A new turbulence model is proposed. The proposed model: a) is able to take into account the anisotropy of the turbulence; b) remove any balance assumption between the production and dissipation of subgrid scale turbulent kinetic energy; c) is able to eliminate the numerical effects produced by the non conservation *a priori* of the resolved kinetic energy. New closure relations for the unknown terms of the subgrid scale viscous dissipation balance equation are proposed. The filtered momentum equations are solved by using a sixth order finite difference scheme. The proposed model is tested for a turbulent channel flow at Reynolds numbers (based on friction velocity and channel half-width) ranging from 395 to 2340.

Keywords: LES, anisotropy, SGS turbulent kinetic energy, two-equation

1 Introduction

Among the most common LES models present in literature are the dynamic Smagorinsky-type SGS models (e.g., Dynamic Smagorinsky Model [4], Dynamic Mixed Model [23], [13], Lagrangian Dynamic Model [11], Dynamic Two-parameter Model [15], in which the generalized SGS turbulent stress tensor, τ_{ij}, is related to the resolved strain-rate tensor by means of a scalar eddy viscosity. It is assumed in these models that the eddy viscosity is a scalar proportional to the cubic root of the generalized SGS turbulent kinetic energy dissipation and that such dissipation is locally and instantaneously balanced by the production of the generalized SGS turbulent kinetic energy (i.e., by the rate of kinetic energy per unit of mass transferred from the large scales, larger than the filter size, to the unresolved ones). Consequently, it is evident that the dynamic Smagorinsky-type SGS models are fraught with four relevant drawbacks. The first drawback is represented by the scalar definition of the eddy viscosity; the second one concerns the local balance assumption of the generalized SGS turbulent kinetic energy production and dissipation, the third drawback is related to the dynamic calculation of the coefficient used to model the eddy viscosity (Smagorinsky coefficient), whilst the fourth drawback is related to the problems arising from the numerical scheme adopted for the simulations of three-dimensional unsteady flows (LES).

The scalar definition (first inconsistency) of the eddy viscosity is equivalent to assuming that the principal axes of the generalized SGS turbulent stress tensor, or the unresolved part of it (represented by the cross and Reynolds terms), are aligned with the principal axes of the resolved strain-rate tensor. This assumption has been disproved by many experimental tests and by DNS, which demonstrate that there is no alignment between the generalized SGS turbulent stress tensor, or the unresolved part of it, and the resolved strain-rate tensor (Tao et al. [19], Meneveau & Katz [10]). Moreover, the eddy viscosity is proportional to the product of two terms, of which the dimensions are, respectively, those of a length and a velocity (Tennekes & Lumley [20]). These terms, which represent, respectively, the turbulence length scales and turbulence velocity scales, are, more generally, second-order tensors of which the product is a fourth-order tensor which represents the eddy viscosity (Monin & Yaglom [12]).

The scalar definition of the eddy viscosity, used in the above-mentioned dynamic Smagorinsky-type SGS models, presupposes the existence of a single turbulence velocity scale and a single turbulence length scale. This is equivalent to assuming that the second-order tensors which represent the turbulence length scales and the turbulence velocity scales are isotropic and that, therefore, the turbulence is isotropic. In this manner, the turbulence anisotropy induced by the continuous transfer of energy from the mean flow towards the turbulent fluctuations, which is generally extremely anisotropic, is not considered. Even though the energy cascade process causes a reduction of the turbulence anisotropy, many authors (Speziale & Gatski [17], Sreenivasan [18]) demonstrated that even in the dissipation range of the smallest turbulent scales, where viscous dissipation occurs, there is a high anisotropy level even at high Reynolds numbers.

The second inconsistency of the Smagorinsky dynamic models is related to the assumption of a local and instantaneous balance between production and dissipation of the generalized SGS turbulent kinetic energy, formulated in the above-mentioned models to obtain the turbulent viscosity expression. This assumption is confirmed statistically and never instantaneously, and only locally at the scales associated with wavenumbers within the inertial subrange, and the latter exists only for isotropic turbulence and at high Reynolds numbers. Moreover, since the dissipation of the generalized SGS turbulent kinetic energy is, by definition, positive, the assumption of local balance implies that also the production of generalized SGS turbulent kinetic energy is positive. However, the assumption that the production is always positive implies that the energy transfer always occurs from the largest to the smallest scales and prevents positive transfers of kinetic energy from the subgrid scales to the resolved ones (backscatter). Since the energy exchange processes between the resolved and unresolved scales generally occur in both directions (forward scatter and back scatter), as has been observed by various authors (Piomelli et al. [14], Horiuti [7]), the assumption that the production of generalized SGS turbulent kinetic energy is always positive does not enable the complexity of the energy exchange processes which characterize the turbulence to be adequately taken into account.

The current dynamic models claim to represent the energy transfer from the smaller to the larger scales (backscatter) by the change to negative values of the Smagorinsky coefficient C_S (which appears in the definition of the eddy viscosity) which is dynamically calculated by means of Germano's procedure [4]. It has, however, been found that when the coefficient C_S assumes negative values the numerical calculation becomes extremely unstable. This instability is due to the long autocorrelation time of the coefficient C_S which, once it becomes negative in some region of the domain, it may remain negative for excessively long periods of time during which the exponential growth of the local velocity fields causes a divergence of the total kinetic energy (Ghosal et al [6]).

The third inconsistency of the dynamic models concerns the calculation of the above mentioned Smagorinsky coefficient C_S. It is calculated with variational methods, (e.g. with a least squares minimization method [8] or Lagrangian method [11]). These methods identify a single value of the scalar coefficient C_S from a system of five independent scalar equations relating the components of the anisotropic part of the generalized SGS turbulent stress tensor to the components of the resolved strain-rate tensor. This procedure does not provide completely acceptable results. Moreover, when simulating confined flows at high Reynolds number, the results of the dynamic procedure are of doubtful reliability in the region close to the wall including both the viscous sublayer and the buffer layer (Sarghini et al. [16]). In this region, the filter width used in the dynamic procedure is larger than most eddies that govern the momentum and energy transfer. Consequently, the dynamic procedure used under these conditions for the calculation of the coefficient C_S is not able to fully account for the local subgrid dissipative processes that affect the entire domain.

The fourth inconsistency of the dynamic mixed models, based on the Smagorinsky closure relation, is connected to the problems arising from the numerical scheme adopted for the simulations of three-dimensional unsteady flows (LES). In the simulations of three-dimensional unsteady flows (that are realized by a high-order finite difference scheme with the divergence form of the convective terms) the resolved kinetic energy is not perfectly conserved because the continuity equation is not perfectly satisfied. Consequently, without an adequate turbulence model, the resolved kinetic energy is destined to rise in long time simulations. In the dynamic mixed models, based on the Smagorinsky closure relation, the calculation of the closure coefficient C_S (by dynamic procedure) is not able to compensate the effects produced by the convective terms (expressed in divergence form in the resolved momentum equation) that are not able to perfectly conserve resolved kinetic energy.

In this paper, a new model for the generalised SGS turbulent stress tensor is proposed in which, in order to adequately account for the anisotropy of both, the turbulence length scales and the turbulence velocity scales, the eddy viscosity is defined as a symmetric fourth order tensor. A mixed formulation is adopted in the model, in which the modified Leonard tensor is calculated explicitly, whilst the unresolved residual part of the tensor τ_{ij} is obtained by the contraction of the eddy viscosity tensor with the resolved strain rate tensor. The principal axes of the generalised SGS turbulent stress tensor τ_{ij} are assumed to be aligned with those of the modified Leonard tensor, in agreement with the assumption of scale similarity [1].

In order to overcome the inconsistencies of the dynamic mixed Smagorinsky - type models and in order to contain the increase of resolved kinetic energy, the turbulent closure relation for the generalised SGS turbulent stress tensor is expressed as a function of the generalised SGS turbulent kinetic energy $E$and of the SGS viscous dissipation ε.

In the proposed model the closure coefficient which appears in the closure relation is uniquely determined without adopting Germano's dynamic procedure. The closure relation at the basis of the proposed Two-Equation Model

(TEM): a) complies with the rule of turbulent closure relations; b) takes into account both the anisotropy of the turbulence velocity scales and of the turbulence length scales; c) removes any assumption of balance between the production and dissipation of the turbulent kinetic energy; d) allows the use of a filter of which the width is not necessarily associated with the wavenumbers lying within the inertial subrange; e) assumes scale similarity in the definition of the second-order tensor representing the turbulence velocity scales; f) guarantees an adequate energy drain from the grid scales to the subgrid scales and guarantees backscatter; g) overcomes the inconsistencies linked to the dynamic calculation of the closure coefficient used in the modelling of the generalized SGS turbulent stress tensor; h) is able to compensate dynamically the increase of resolved kinetic energy (produced by the numerical discretization of the divergence form of the convective term in momentum equations) and then allow the large eddy simulation of three-dimensional unsteady flows, also for long time simulations.

A sixth order accurate scheme for a non-uniform staggered grid with good conservation properties is adopted: the proposed scheme conserves mass and momentum; the non-conservation of kinetic energy is weak. It is a function of a commutation error which is very small for smoothly varying meshes (Vasilyev [21]).

2 Energy conservation property of the numerical scheme in Large Eddy Simulation

The fourth inconsistency of the dynamic mixed models (based on the Smagorinsky closure relation) is connected to the problems arising from the numerical scheme adopted for the simulations of three - dimensional unsteady flows (LES).

The three - dimensional unsteady flows simulations require numerical schemes with a high order of accuracy: a low order of accuracy of centred finite difference schemes introduces an anti-dissipative factor, which reduces the ability of the generalised SGS turbulent stress tensor to represent the kinetic energy transfer from the resolved scales to the unresolved ones, with an increase of the resolved kinetic energy.

The numerical scheme, besides being accurate, must fulfil the conservation requirement.

Conservation properties of the mass, the momentum and the kinetic energy equations, for incompressible flows, are regarded as analytical requirements for a proper set of discrete equations. Consider the following governing equation for the scalar quantity ϕ:

$$\frac{\partial \phi}{\partial t} + {}^{1}Q\left(\phi\right) + {}^{2}Q\left(\phi\right) + {}^{3}Q\left(\phi\right) + ... = 0 \tag{1}$$

the term ${}^{k}Q\left(\phi\right)$ is conservative (conserves the volume integral of ϕ over the whole domain in periodic field) if it can be written in divergence form

$${}^{k}Q\left(\phi\right) = \frac{\partial \left({}^{k}F_j\right)}{\partial x_j} \tag{2}$$

Note that mass is conserved *a priori* since the continuity equation appears in divergence form.

The convective term of the momentum equations for an incompressible flow can be expressed in divergence and advective form:

$$\left(Div.\right)_i = \frac{\partial u_j u_i}{\partial x_j}; \quad \left(Adv.\right)_i = u_j \frac{\partial u_i}{\partial x_j} \tag{3}$$

The relation between the divergence and the advective form of the convective term is given by:

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j u_i}{\partial x_j} - u_i \frac{\partial u_j}{\partial x_j} \tag{4}$$

The advective form of the convective term conserves momentum (i.e. the convective term volume integral over the whole domain, in periodic field, does not modify the global volume integral of momentum) only if the continuity equation (for an incompressible flow) is perfectly satisfied.

The convective term of the momentum equation is conservative *a priori* if it is written in divergence form.

This definition of the conservation *a priori* indicates the property of conserving momentum independently of the modalities by which the continuity equation is satisfied.

The governing equation for the kinetic energy, $K = u_i u_i/2$, can be developed by taking the vector dot product of the velocity and the momentum equation,

$$u_i \left(\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial \sigma_{ij}}{\partial x_j}\right) = 0 \tag{5}$$

where p is the pressure divided by the constant density, and σ_{ij} is the viscous stress. In the above equation the convective term can be rewritten in the following form, corresponding to that in the momentum equation,

$$u_i \frac{\partial u_j u_i}{\partial x_j} = \frac{\partial u_j u_i u_i/2}{\partial x_j} + \frac{1}{2} u_i u_i \frac{\partial u_j}{\partial x_j} \tag{6}$$

This term is composed by two parts: the first is in conservative form and the second involves the continuity equation. The convective term (expressed in divergence form in the momentum equation) conserves *a priori* momentum but does not conserve *a priori* kinetic energy: in fact equation (6) shows how the continuity equation is involved in the kinetic energy conservation property of the convective terms that are expressed in divergence form. In other words, kinetic energy volume integral is conserved, in a periodic field, (by the divergence form of the convective term) only when the continuity equation is perfectly satisfied.

The passage from the previous analytical considerations to the effect that they produce on numerical simulations imposes a reflection on the following statement:

the continuity equation cannot be perfectly satisfied by numerical simulation.

In the simulations of three-dimensional unsteady flows (LES) (that are realized by a high-order finite difference scheme with the divergence form of the convective terms) the resolved kinetic energy is not perfectly conserved because the continuity equation is not perfectly satisfied. Consequently, in LES, the numerical discretization of the divergence form of the convective term in the momentum equation introduce a numerical error associated with a production of resolved kinetic energy.

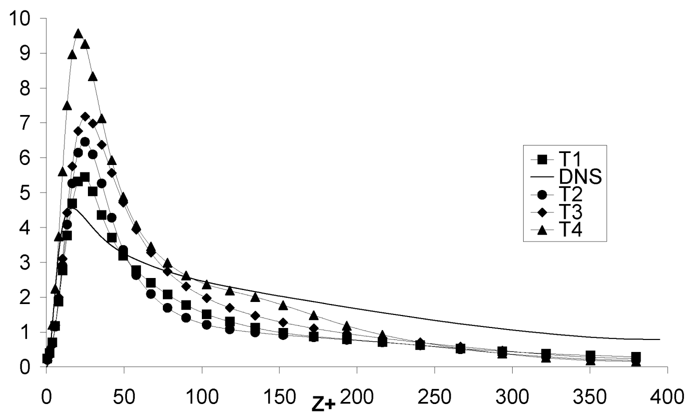

Figure 1: Profiles of resolved kinetic energy Reynolds averaged over successive intervals of time (T1, T2, T3, T4). Simulation performed by using the turbulence model of Zang et al. [23]. Channel flow, Re*=395.

Without a turbulence model that is able to dissipate the above mentioned kinetic energy production, the resolved kinetic energy is destined to rise in long time simulations.

In the dynamic mixed models based on the Smagorinsky closure relation, the calculation of the closure coefficient C_S (by dynamic procedure) is not able to compensate the effects produced by the convective terms (expressed in divergence form in the resolved momentum equation) that are not able to perfectly conserve resolved kinetic energy.

In order to verify the above mentioned inconsistency of the dynamic mixed models based on the Smagorinsky closure relation, simulations of a turbulent channel flow at Re*=395 (Re* is the friction-velocity-based Reynolds number) have been performed, by a sixth-order staggered finite difference scheme proposed by Vasilyev [21]. The generalised SGS turbulent stress tensor has been calculated by means of the mixed dynamic model of Zang et al. [23]. The resolved kinetic energy has been averaged over time intervals greater than the integral turbulent time scale. In Figure 1 the over time averaged resolved kinetic energy profiles are shown. From the figure it is possible to deduce that the resolved kinetic energy increases.

In this paper it is demonstrated that, in order to contain the increase of resolved kinetic energy, the turbulent closure relation for the generalised SGS turbulent stress tensor must be expressed directly as a function of the generalised SGS turbulent kinetic energy E and of the SGS viscous dissipation ε.

The generalised SGS turbulent kinetic energy and the SGS viscous dissipation are unknown quantities that are calculated by solving the relative balance equations. In these equations there are unknown terms that are calculated by dynamic procedures: it is shown that the dynamic procedures for the calculation of the closure coefficients of the production and dissipation terms of the SGS viscous dissipation balance equation are able to compensate dynamically the increase of resolved kinetic energy (produced by the numerical discretization of the divergence form of the convective term in the momentum equation) and then allow the large eddy simulation of three-dimensional unsteady flows, also for long time simulations.

3 The turbulence model

The generalised SGS turbulent stress tensor, τ_{ij}, can be split into three tensors

$$\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j = L_{ij}^m + C_{ij}^m + R_{ij}^m \tag{7}$$

where u_i is the ith component of the instantaneous velocity, the overbar represents the application of the grid filtering operator G,

$$L_{ij}^m = \overline{\overline{u}_i \overline{u}_j} - \overline{\overline{u}}_i \overline{\overline{u}}_j, \quad C_{ij}^m = \overline{\overline{u}_i u_j'} - \overline{\overline{u}}_i \overline{u_j'} + \overline{u_i' \overline{u}_j} - \overline{u_i'} \overline{\overline{u}}_j, \quad R_{ij}^m = \overline{u_i' u_j'} - \overline{u_i'}\, \overline{u_j'} \tag{8}$$

are, respectively, the so-called modified Leonard tensor, modified cross tensor and modified Reynolds tensor and $u_i' = u_i - \overline{u}_i$.

In dynamic Smagorinsky-type mixed models the modified cross tensor C_{ij}^m and the modified Reynolds tensor R_{ij}^m are related to the resolved strain rate tensor $\overline{S}_{ij}$ by means of scalar eddy viscosity ν_T. The alignment assumption between the tensor $C_{ij}^m + R_{ij}^m$ and $\overline{S}_{ij}$ is not experimentally verified [10], [19].

In order to remove this assumption and take into account the anisotropy of the unresolved scales of turbulence, we express the generalised SGS turbulent stress tensor in the form:

$$\tau_{ij} = L_{ij}^m - 2\nu_{ijmn}\overline{S}_{mn} \tag{9}$$

The eddy viscosity is expressed in the above equation by a fourth order tensor proportional to the product of a second-order tensor, b_{ij}, which represents the turbulence length scales, and a and a second-order tensor, d_{mn}, which represents the turbulence length scales, according to the equation

$$\nu_{ijmn} = Cb_{ij}d_{mn}, \text{ where } b_{ij} = b_{ji} \text{ and } d_{mn} = d_{nm} \tag{10}$$

The expression of the eddy viscosity in terms of a fourth order tensor enables the anisotropic character of the turbulence to be fully represented, since it assume neither the existence of a single turbulence velocity scale nor a single turbulence length scale, as is found in models in which the viscosity is expressed as a scalar.

The second order tensor that represents the turbulence velocity scales is defined as

$$b_{ij} = \sqrt{E}L_{ij}^m \Big/ L_{kk}^m \tag{11}$$

In which E is the generalised SGS turbulent kinetic energy. In this equation it is assumed that the second-order tensor b_{ij} is proportional to the square root of the generalized SGS turbulent kinetic energy E and that it is aligned with the modified Leonard tensor. In this manner it is assumed that the anisotropy of the unresolved turbulence velocity scales, expressed by b_{ij}, is equal to the anisotropy of the smallest resolved scales, associated with the modified Leonard tensor L_{ij}^m. This assumption is based on scale similarity, according to which the scales that are contiguous in the wavenumber space have strict dynamic analogies related to the energy exchange processes, which occur between them. In order to take into account the complexities of the phenomena linked to the anisotropy of the unresolved turbulence length scales, the anisotropy of the tensor d_{mn} is related to the anisotropy of the filter used. Consequently, the second order tensor d_{mn} is defined as

$$d_{mn} = \Delta_m\Delta_n \Big/ \sqrt[3]{\Delta_1\Delta_2\Delta_3} \tag{12}$$

In which Δ_m is the vector of which the components are the filters dimensions in the three coordinates directions.

With (10), (11) and (12) the generalized SGS turbulent stress tensor takes the form:

$$\tau_{ij} = L_{ij}^m - 2Cd_{mn}\bar{S}_{mn}\sqrt{E}L_{ij}^m \Big/ L_{kk}^m \tag{13}$$

which may be rewritten in the more compact form

$$\tau_{ij} = (1 + r)\, L_{ij}^m, \text{ where } r = -2Cd_{mn}\bar{S}_{mn}\sqrt{E} \Big/ L_{kk}^m \tag{14}$$

The coefficient r in (14) is uniquely determined by using the existing relation between the generalized SGS turbulent kinetic energy and the generalized SGS turbulent stress tensor. By definition, the generalized SGS turbulent kinetic energy is equal to half the trace of the generalized SGS turbulent stress tensor

$$E = \tau_{kk}/2 \tag{15}$$

Equation (14) and (15) gives

$$\tau_{kk} = (1 + r)L_{kk}^m \quad , \tag{16}$$

from which we have

$$r = \left(2E - L_{kk}^m\right)/L_{kk}^m \quad . \tag{17}$$

Introducing (17) into (14) gives:

$$\tau_{ij} = \left(1 + \frac{2E - L_{kk}^m}{L_{kk}^m}\right) L_{ij}^m = \left(\frac{2E}{L_{kk}^m}\right) L_{ij}^m. \tag{18}$$

The generalized SGS turbulent stress tensor is expressed in equations (18) by means of a tensor of which the principal axes are aligned with those of the modified Leonard tensor L_{ij}^m.

The closure relation expressed by equations (18) is obtained without any assumption of local balance between production and dissipation of the generalized SGS turbulent kinetic energy. The assumption of local balance between the production and dissipation of the generalized SGS turbulent kinetic energy is valid, in fact, (in a statistical sense) only for homogeneous and isotropic turbulence and within the inertial subrange, which only occurs at high Reynolds numbers and at high wavenumbers.

Given that the assumption of local balance between production and dissipation of the generalized SGS turbulent kinetic energy only occurs in the inertial subrange (when this exists), the above-mentioned assumption requires the use of spatial filters of which the dimensions are associated with wavenumbers belonging to the inertial subrange itself. With the increase in the Reynolds number the inertial subrange occurs at increasingly high wavenumbers and, therefore, at increasingly small turbulence scales. Consequently, the assumption of local balance between production and dissipation of the generalized

SGS turbulent kinetic energy requires the use of finer grids as the Reynolds number increases. The closure relation expressed by (18), removing the above-mentioned balance assumption, does not require the use of spatial filters (and, therefore, calculation grids) of dimensions associated with wavenumbers falling in the inertial subrange and may therefore be used with coarser calculation grids.

The sequence of equations (9)-(18) demonstrates that the definition of the eddy viscosity in terms of a fourth-order tensor and the definition of the tensor of the turbulence velocity scales aligned with the modified Leonard tensor are equivalent to the assumption of scale similarity. This enables the formulation of a closure relation for the generalized SGS turbulent stress tensor similar to that of many scale similarity models derived from the Bardina model [1]. It can, therefore, easily be seen that the existence of scale similarity between the turbulent structures associated with contiguous scales is in line with the definition of a fourth-order turbulent viscosity tensor, which removes the assumption of local isotropy. Consequently, the formulation of the generalized SGS turbulent stress tensor defined by the sequence of equations (9)-(18) may be considered applicable in LES with filter width falling into the range of wavenumbers greater than the wavenumber corresponding to the maximum turbulent kinetic energy.

On the other hand, in the range of wavenumbers that are below the latter, even though the strong anisotropy of the turbulent structures suggests the use of the fourth-order eddy viscosity tensor, the scale similarity assumption may not be reasonably formulated. In this range, in fact, the part of the turbulent kinetic energy produced by the largest unresolved eddies is high and consequently the above-mentioned turbulent structures may not entirely represent the energy dissipation process (a basic assumption of scale similarity models).

The generalised turbulent kinetic energy E is calculated by solving its balance equation

$$\frac{DE}{Dt} = -\frac{1}{2}\frac{\partial \tau(u_k, u_k, u_m)}{\partial x_m} - \tau\left(u_m, u_k\right)\frac{\partial \bar{u}_k}{\partial x_m} - \frac{\partial \tau\left(p, u_m\right)}{\partial x_m} + \nu \frac{\partial^2 E}{\partial x_m \partial x_m} - \nu\tau\left(\frac{\partial u_k}{\partial x_m}, \frac{\partial u_k}{\partial x_m}\right) \quad (19)$$

The symbols $\tau(f, g)$ and $\tau(f, g, h)$ represent the generalized second and third-order central moments (Germano [5]) related to the generic quantities f , g and h, namely

$$\tau(f, g) = \overline{f\,g} - \overline{f}\,\overline{g} \quad (20)$$

$$\tau(f, g, h) = \overline{f\,g\,h} - \overline{f}\,\overline{g}\,\overline{h} - \overline{f}\,\tau(g, h) - \overline{g}\,\tau(f, h) - \overline{h}\,\tau(f, g) \quad (21)$$

Equation (19) is form invariant under Euclidean transformation of the frame and frame indifferent (Gallerano et al. [3]).

Consequently, the modelled balance equation of E must be form-invariant and frame-indifferent, like the exact balance equation of E.

The sum of the 1^{st} and 3^{rd} term of the right-hand side of equation (19) expresses the turbulent diffusion of the generalised SGS turbulent kinetic energy, for which the following closure relation is proposed:

$$\frac{1}{2}\tau\left(u_i,u_i,u_k\right)+\tau\left(p,u_k\right)=D\sqrt{E}\overline{\Delta}\frac{\partial E}{\partial x_k} \tag{22}$$

The scalar coefficient D is dynamically calculated by means of the identity:

$$\begin{aligned}\frac{1}{2}T\left(u_i,u_i,u_k\right)+T\left(p,u_k\right)-\left\langle\frac{1}{2}\tau\left(u_i,u_i,u_k\right)+\tau\left(p,u_k\right)\right\rangle=\\+\frac{1}{2}\left\langle\overline{u}_i\,\overline{u}_i\,\overline{u}_k\right\rangle-\frac{1}{2}\left\langle\overline{u}_i\right\rangle\left\langle\overline{u}_i\right\rangle\left\langle\overline{u}_k\right\rangle-\frac{1}{2}\left\langle\overline{u}_k\right\rangle T\left(u_i,u_i\right)+\frac{1}{2}\left\langle\overline{u}_k\tau\left(u_i,u_i\right)\right\rangle\\-\left\langle\overline{u}_i\right\rangle T\left(u_i,u_k\right)+\left\langle\overline{u}_i\tau\left(u_i,u_k\right)\right\rangle+\left\langle\overline{p}\overline{u}_k\right\rangle-\left\langle\overline{p}\right\rangle\left\langle\overline{u}_k\right\rangle\end{aligned} \tag{23}$$

where the angular brackets represent the application of a filtering operator F, with characteristic dimension, $\overline{\Delta}^T$, which is double that the grid filter one and

$$T\left(f,g\right)=\left\langle\overline{fg}\right\rangle-\left\langle\overline{f}\right\rangle\left\langle\overline{g}\right\rangle \tag{24}$$

$$T\left(f,g,h\right)=\left\langle\overline{fgh}\right\rangle-\left\langle\overline{f}\right\rangle\left\langle\overline{g}\right\rangle\left\langle\overline{h}\right\rangle-\left\langle\overline{f}\right\rangle T\left(g,h\right)-\left\langle\overline{g}\right\rangle T\left(f,h\right)-\left\langle\overline{h}\right\rangle T\left(f,g\right) \tag{25}$$

are, respectively, the generalised second and third order central moments relative to test-scale filtering operator *G1=FG.*_

The second-order generalised central moments relative to the test and is linked to the second-order generalised central moments relative to the grid filter by means of the Germano identity [4]:

$$T\left(f,g\right)=\left\langle\tau\left(f,g\right)\right\rangle+\left\langle\overline{f}\,\overline{g}\right\rangle-\left\langle\overline{f}\right\rangle\left\langle\overline{g}\right\rangle \tag{26}$$

For the generalised central moments, relative to the test filter, on the left - hand side of (23) we propose the following closure relation

$$\frac{1}{2}T\left(u_i,u_i,u_k\right)+T\left(p,u_k\right)=D\sqrt{E^T}\overline{\Delta}^T\frac{\partial E^T}{\partial x_k} \tag{27}$$

where E^T is the generalised SGS turbulent kinetic energy relative to the test filter. By introducing equations (22) and (27) into the left-hand side of (23), by introducing equations (15) and (18) into the right-hand side of (23) and by using (26), the closure coefficient D is obtained.

By introducing (18) and (22) into (19), the proposed modelled form of the generalised SGS turbulent kinetic energy is obtained:

$$\frac{DE}{Dt}=-\frac{\partial}{\partial x_k}\left(D\sqrt{E}\overline{\Delta}\frac{\partial E}{\partial x_k}\right)-\left(\frac{2E}{L_{qq}^m}\right)L_{mk}^m\frac{\partial\bar{u}_k}{\partial x_m}+\nu\frac{\partial^2 E}{\partial x_m\partial x_m}-\varepsilon \tag{28}$$

The last term on the right-hand side of equation (19) is the generalised SGS viscous dissipation

$$\varepsilon = \nu\tau\left(\frac{\partial u_k}{\partial x_m}, \frac{\partial u_k}{\partial x_m}\right) \tag{29}$$

In the proposed turbulence model the generalised SGS viscous dissipation is calculated by solving its balance equation.

The exact ε balance equation, expressed in terms of the generalized central moments, takes the form:

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial t} + \frac{\partial \overline{u}_k \varepsilon}{\partial x_k} - \nu\frac{\partial^2 \varepsilon}{\partial x_k \partial x_k} + \nu\frac{\partial}{\partial x_k}\tau\left(u_k, \frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\tau\left(u_k, \frac{\partial u_i}{\partial x_j}\right)\right) + 2\nu\frac{\partial}{\partial x_k}\tau\left(\frac{\partial u_k}{\partial x_i}, \frac{\partial p}{\partial x_i}\right)\\
&-2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \tau\left(u_i, u_k\right)}{\partial x_j}\right) + 2\nu\tau\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_k}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial \overline{u}_k}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_i}{\partial x_j}\right) + 2\nu\frac{\partial \overline{u}_i}{\partial x_k}\tau\left(\frac{\partial u_k}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial \overline{u}_i}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_k}{\partial x_j}\right) + 2\nu\frac{\partial \tau\left(u_i, u_k\right)}{\partial x_j}\frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\\
&+2\nu^2\tau\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}, \frac{\partial^2 u_i}{\partial x_j \partial x_k}\right) = 0
\end{aligned} \tag{30}$$

This equation is obtained from the Navier-Stokes equation and the filtered Navier-Stokes equation.

In the Gallerano et al. model [3], the 5^{th} and 7^{th} terms of the above mentioned equation are added together. It is easy to demonstrate that the 11^{th} term and the sum between the 5^{th} and the 7^{th} terms are the representations, in an inertial frame, of two objective but frame-dependent zero-order tensors. The 11^{th} term and the above sum are unknown quantities that must be modelled. The modelled expression of the sum of the two terms (5^{th} and 7^{th}) proposed in [3] is not correct, since they represent quantities with different physical meanings; the 5^{th} is a convective transport term, while the 7^{th} represents a dispersive transport term.

Furthermore, there is an evident contradiction in [3]: the modelled expressions in [3] of the 11^{th} term and the sum between the 5^{th} and 7^{th} term are objective quantities and frame-indifferent, while the expressions of the corresponding terms that appear in the exact ε balance equation are objective quantities but frame-dependent. In this paper the above contradictions are overcome.

The sum of the 4^{th} and 6^{th} terms of equation (30) are modelled as follow

$$\nu \frac{\partial}{\partial x_k} \tau \left(u_k, \frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j} \right) + 2\nu \frac{\partial}{\partial x_k} \tau \left(\frac{\partial u_k}{\partial x_i}, \frac{\partial p}{\partial x_i} \right) = C_{F_\varepsilon} \frac{E^2}{\varepsilon} \frac{L^m_{kl}}{L^m_{jj}} \frac{\partial \varepsilon}{\partial x_l} \tag{31}$$

The closure coefficient C_{F_ε} is calculated dynamically, by means of the identity:

$$\begin{aligned} &\nu \left[T \left(u_k, \frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j} \right) + 2T \left(\frac{\partial u_k}{\partial x_i}, \frac{\partial p}{\partial x_i} \right) \right] \\ &-\nu \left\langle \tau \left(u_k, \frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j} \right) + 2\tau \left(\frac{\partial u_k}{\partial x_i}, \frac{\partial p}{\partial x_i} \right) \right\rangle = \\ &-\nu \left\langle \overline{u}_k \right\rangle \left\langle \frac{\partial \overline{u}_i}{\partial x_j} \right\rangle \left\langle \frac{\partial \overline{u}_i}{\partial x_j} \right\rangle - \left\langle \overline{u}_k \right\rangle \nu T \left(\frac{\partial u_r}{\partial x_m}, \frac{\partial u_r}{\partial x_m} \right) \\ &-2\nu \left\langle \frac{\partial \overline{u}_i}{\partial x_j} \right\rangle T \left(u_k, \frac{\partial u_i}{\partial x_j} \right) + \nu \left\langle \overline{u}_k \frac{\partial \overline{u}_i}{\partial x_j} \frac{\partial \overline{u}_i}{\partial x_j} \right\rangle \\ &+ \left\langle \overline{u}_k \tau \left(\frac{\partial u_r}{\partial x_m}, \frac{\partial u_r}{\partial x_m} \right) \right\rangle + 2\nu \left\langle \frac{\partial \overline{u}_i}{\partial x_j} \tau \left(u_k, \frac{\partial u_i}{\partial x_j} \right) \right\rangle \\ &+2\nu \left\langle \frac{\partial \overline{u}_k}{\partial x_i} \frac{\partial \overline{p}}{\partial x_i} \right\rangle - 2\nu \left\langle \frac{\partial \overline{u}_k}{\partial x_i} \right\rangle \left\langle \frac{\partial \overline{p}}{\partial x_i} \right\rangle \end{aligned} \tag{32}$$

On the right-hand side of equation (32) the unknown second-order generalised central moment, $\tau(u_k, \partial u_i / \partial x_j)$, and the corresponding second-order generalised central moment relative to the test filter, $T(u_k, \partial u_i / \partial x_j)$, appear.

By assuming the scale similarity assumption, the above mentioned generalised central moment relative to the grid filter becomes:

$$\tau(u_k, \partial u_i / \partial x_j) = \tau(\overline{u}_k, \partial \overline{u}_i / \partial x_j) \left(\tau(u_i, \partial u_i / \partial x_i) / \tau(\overline{u}_m, \partial \overline{u}_m / \partial x_m) \right) \tag{33}$$

where $\tau(u_i, \partial u_i / \partial x_i)$ is an unknown second-order generalised central moment that must be modelled. The following closure relations for this quantity and for the corresponding second-order generalised central moment relative to the test filter are proposed:

$$\tau \left(u_i, \frac{\partial u_i}{\partial x_i} \right) = C_{T_\varepsilon} \frac{E}{\varepsilon} \frac{\partial \varepsilon}{\partial x_q} \delta_q, \qquad T \left(u_i, \frac{\partial u_i}{\partial x_i} \right) = C_{T_\varepsilon} \frac{E^T}{\varepsilon^T} \frac{\partial \varepsilon^T}{\partial x_q} \delta_q \tag{34}$$

where $\varepsilon^T$is the generalised SGS viscous dissipation relative to the test filter.

The closure coefficient C_{T_ε} is calculated by means of the following identity:

$$T \left(u_i, \frac{\partial u_i}{\partial x_i} \right) - \left\langle \tau \left(u_i, \frac{\partial u_i}{\partial x_i} \right) \right\rangle = \left\langle \overline{u}_i \frac{\partial \overline{u}_i}{\partial x_i} \right\rangle - \left\langle \overline{u}_i \right\rangle \left\langle \frac{\partial \overline{u}_i}{\partial x_i} \right\rangle \tag{35}$$

By introducing $(34)_1$ and $(34)_2$ into the left-hand side of equation (35), the closure coefficient C_{T_ε} is obtained.

For the generalised central moments, relative to the test filter, on the left-hand side of (31) the proposed closure relation is

$$\nu T\left(u_k, \frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right) + 2\nu T\left(\frac{\partial u_k}{\partial x_i}, \frac{\partial p}{\partial x_i}\right) = C_{F_\varepsilon}\frac{\left(E^T\right)^2}{\varepsilon^T}\frac{L_{kl}^{mT}}{L_{jj}^{mT}}\frac{\partial \varepsilon^T}{\partial x_l} \tag{36}$$

where L_{ik}^{mT} is the modified Leonard tensor relative to the test filter.

By introducing the closure relations (31) and (36) into the left-hand side of equation (32), by introducing (33) and (34) into the right-hand side of equation (32) and by using (26), the closure coefficient C_{F_ε} is obtained.

The 5^{th} term of equation (30) is calculated by using the scale similarity hypothesis, expressed by equation (33), and by modelling only the unknown term on the right-hand side of (33) by means of the closure relation (34). The closure coefficient on the right-hand side of (34) is dynamically calculated by means of equation (35). The resulting closure relation for the 5^{th} term of (30) is:

$$2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\tau\left(u_k, \frac{\partial u_i}{\partial x_j}\right)\right) = 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\left(C_{T_\varepsilon}\frac{E}{\varepsilon}\frac{\partial \varepsilon}{\partial x_q}\delta_q\frac{\tau\left(\overline{u}_k, \frac{\partial \overline{u}_i}{\partial x_j}\right)}{\tau\left(\overline{u}_n, \frac{\partial \overline{u}_n}{\partial x_n}\right)}\right)\right) \tag{37}$$

As it is easy to demonstrate, the proposed expression for the 5^{th} term of equation (30) results dependent on the frame of reference, under Euclidean transformations of the frame, the same manner of the no-modelled term.

The 7^{th} term of (30) is related exclusively to the resolved velocity field and the generalised SGS turbulent stress tensor. It is calculated by means of the closure relation (18) for the generalised SGS turbulent stress tensor. The resulting closure relation for the 7^{th} term of (30) is:

$$2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \tau\left(u_i, u_k\right)}{\partial x_j}\right) = 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{2E}{L_{qq}^m}L_{ik}^m\right)\right) \tag{38}$$

Under a Euclidean transformation of the frame, the modelled expression of the 7^{th} term of equation (30) results dependent on the frame of reference the same manner of the no-modelled term.

The 8^{th} term of equation (30) represents a production term of ε. For this tensor the proposed closure relation is

$$2\nu\tau\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_k}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right) = C_{P_\varepsilon}\varepsilon\left(-L_{ij}^m\overline{S}_{ij}\right)\Big/L_{kk}^m \tag{39}$$

The closure coefficient C_{P_ε} is dynamically calculated by means of the identity:

$$2\nu T\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_k}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right) - 2\nu\left\langle\tau\left(\frac{\partial u_i}{\partial x_k}, \frac{\partial u_k}{\partial x_j}, \frac{\partial u_i}{\partial x_j}\right)\right\rangle =$$

$$\begin{aligned}
&2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_k}\tau\left(\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)\right\rangle+2\nu\left\langle\frac{\partial\overline{u}_k}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j}\right)\right\rangle\\
&+2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)\right\rangle-2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_k}\right\rangle T\left(\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)\\
&-2\nu\left\langle\frac{\partial\overline{u}_k}{\partial x_j}\right\rangle T\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j}\right)-2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_j}\right\rangle T\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)\\
&+2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_k}\frac{\partial\overline{u}_k}{\partial x_j}\frac{\partial\overline{u}_i}{\partial x_j}\right\rangle-2\nu\left\langle\frac{\partial\overline{u}_i}{\partial x_k}\right\rangle\left\langle\frac{\partial\overline{u}_k}{\partial x_j}\right\rangle\left\langle\frac{\partial\overline{u}_i}{\partial x_j}\right\rangle
\end{aligned}\tag{40}$$

The unknown generalised second-order central moments in (38) are expressed in terms of resolved variables by assuming the scale similarity assumption.

In particular, by assuming the scale similarity assumption, the unknown generalised second order central moment in the 3^{rd} term of the right-hand side of (40) becomes:

$$\tau\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)=\tau\left(\frac{\partial\overline{u}_i}{\partial x_k},\frac{\partial\overline{u}_k}{\partial x_j}\right)\left(\tau\left(\frac{\partial u_m}{\partial x_s}\frac{\partial u_m}{\partial x_s}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_n},\frac{\partial\overline{u}_q}{\partial x_n}\right)\right)\tag{41}$$

which, for (29), is equivalent to

$$\tau\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)=\frac{\varepsilon}{\nu}\left(\tau\left(\frac{\partial\overline{u}_i}{\partial x_k},\frac{\partial\overline{u}_k}{\partial x_j}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_n},\frac{\partial\overline{u}_q}{\partial x_n}\right)\right)\tag{42}$$

Analogously, the unknown generalised second order central moments in the 1^{st} and 2^{nd} term of the right-hand side of (40) are calculated by means of expressions similar to equation (42).

For the generalised third-order central moment, relative to the test filter, on the left-hand side of (40) the proposed closure relation is

$$2\nu T\left(\frac{\partial u_i}{\partial x_k},\frac{\partial u_k}{\partial x_j},\frac{\partial u_i}{\partial x_j}\right)=C_{P_\varepsilon}\varepsilon^T\left(-L_{ij}^{mT}\overline{S}_{ij}^T\right)\Big/L_{kk}^{mT}\tag{43}$$

where $\overline{S}_{ij}^T$ is the resolved strain rate tensor relative to the test filter.

By introducing the closure relations (39) and (43) into the left-hand side of equation (40), by introducing the closure relation (42) into the right-hand side of (40) and by using (26), the closure coefficient C_{P_ε} is obtained.

The unknown quantities in the 9^{th}, 10^{th} and 11^{th} term of equation (30) are calculated, by using the scale similarity assumption, by means of equations (41) and (42). In particular, for the 11^{th} term of (30) we obtain:

$$2\nu\frac{\partial\overline{u}_i}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k},\frac{\partial u_k}{\partial x_j}\right)=2\varepsilon\frac{\partial\overline{u}_i}{\partial x_j}\left(\tau\left(\frac{\partial\overline{u}_i}{\partial x_k},\frac{\partial\overline{u}_k}{\partial x_j}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_n},\frac{\partial\overline{u}_q}{\partial x_n}\right)\right)\tag{44}$$

As it is easy to demonstrate, the proposed expression for the 11^{th} term of equation (30) results dependent on the frame of reference, under a Euclidean transformation of the frame, the same manner of the no-modelled term.

The 12^{th} term of equation (30) is calculated by means of the closure relation (18) for the generalised SGS turbulent stress tensor.

The last term of equation (30) represents the destruction of ε and is modelled as follow

$$2\nu^2\tau\left(\frac{\partial^2 u_i}{\partial x_j\partial x_k},\frac{\partial^2 u_i}{\partial x_j\partial x_k}\right)=C_{D_\varepsilon}\frac{\varepsilon^2}{E} \tag{45}$$

where the closure coefficient $C_{D_\varepsilon}$is calculated by the identity:

$$2\nu^2T\left(\frac{\partial^2 u_i}{\partial x_j\partial x_k},\frac{\partial^2 u_i}{\partial x_j\partial x_k}\right)-\left\langle 2\nu^2\tau\left(\frac{\partial^2 u_i}{\partial x_j\partial x_k},\frac{\partial^2 u_i}{\partial x_j\partial x_k}\right)\right\rangle=$$
$$2\nu^2\left\langle\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_k}\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_k}\right\rangle-2\nu^2\left\langle\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_k}\right\rangle\left\langle\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_k}\right\rangle \tag{46}$$

The following closure relation is proposed for the 1^{st} term on the left of (46)

$$2\nu^2T\left(\frac{\partial^2 u_i}{\partial x_j\partial x_k},\frac{\partial^2 u_i}{\partial x_j\partial x_k}\right)=C_{D_\varepsilon}\left(\varepsilon^T\right)^2\Big/E^T \tag{47}$$

By introducing (45) and (47) into (46) the closure coefficient $C_{D_\varepsilon}$is obtained.

The final modelled form of equation (30) is

$$\frac{\partial\varepsilon}{\partial t}+\frac{\partial\overline{u}_k\varepsilon}{\partial x_k}-\nu\frac{\partial^2\varepsilon}{\partial x_k\partial x_k}+\frac{\partial}{\partial x_k}C_{F_\varepsilon}\frac{E^2}{\varepsilon}\frac{L^m_{kl}}{L^m_{jj}}\frac{\partial\varepsilon}{\partial x_l}$$
$$+2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial\overline{u}_i}{\partial x_j}\left(C_{T_\varepsilon}\frac{E}{\varepsilon}\frac{\partial\varepsilon}{\partial x_q}\delta_q\tau\left(\overline{u}_k,\frac{\partial\overline{u}_i}{\partial x_j}\right)\Big/\tau\left(\overline{u}_n,\frac{\partial\overline{u}_n}{\partial x_n}\right)\right)\right)$$
$$-2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial\overline{u}_i}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{2E}{L^m_{qq}}L^m_{ik}\right)\right)-C_{P_\varepsilon}\frac{\varepsilon\left(-L^m_{ij}\overline{S}_{ij}\right)}{L^m_{kk}}$$
$$+2\nu\frac{\partial\overline{u}_i}{\partial x_j}\frac{\varepsilon}{\nu}\tau\left(\frac{\partial\overline{u}_i}{\partial x_k},\frac{\partial\overline{u}_i}{\partial x_j}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_n},\frac{\partial\overline{u}_q}{\partial x_n}\right)$$
$$+2\nu\frac{\partial\overline{u}_i}{\partial x_k}\frac{\varepsilon}{\nu}\tau\left(\frac{\partial\overline{u}_k}{\partial x_j},\frac{\partial\overline{u}_i}{\partial x_j}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_n},\frac{\partial\overline{u}_q}{\partial x_n}\right)$$
$$+2\nu\frac{\partial\overline{u}_i}{\partial x_j}\frac{\varepsilon}{\nu}\tau\left(\frac{\partial\overline{u}_i}{\partial x_k},\frac{\partial\overline{u}_k}{\partial x_j}\right)\Big/\tau\left(\frac{\partial\overline{u}_q}{\partial x_s},\frac{\partial\overline{u}_s}{\partial x_q}\right)$$
$$+2\nu\frac{\partial}{\partial x_j}\left(\frac{2E}{L^m_{nn}}L^m_{ik}\right)\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_k}+C_{D_\varepsilon}\frac{\varepsilon^2}{E}=0 \tag{48}$$

The proposed modelled balance equation of the SGS viscous dissipation is form-invariant under Euclidean transformations of the frame and has the same dependence on the frame as the exact equation (the demonstration is omitted for the sake of brevity).

4 The numerical scheme

For the simulation of the unsteady three-dimensional turbulent flow it is very important to control the dissipation produced by the numerical scheme. The numerical dissipation removes energy from the dynamically important small-scale eddies; for this reason unsteady, three-dimensional turbulent simulations are much less tolerant of numerical dissipation (Morinishi et al [13]), consequently, the numerical scheme must be accurate. Morinishi et al [13] derived the general family of fully conservative higher order accurate finite difference schemes for uniform staggered grids. The extension of the scheme, suggested by Morinishi et al., to non uniform meshes produces a fourth order accurate finite difference scheme that is not fully conservative. Vasilyev [21] generalized the high order schemes of Morinishi et al. to non-uniform meshes by preserving the symmetries of the uniform mesh case, obtaining numerical schemes with good conservation properties. The schemes of Vasilyev do not simultaneously conserve mass, momentum and kinetic energy. However, depending on the form of the convective term, conservation of either momentum or energy in addition to mass can be achieved. In this paper we produce a sixth order accurate scheme for a non-uniform staggered grid with good conservation properties: the proposed scheme conserves mass and momentum; the non-conservation of kinetic energy is weak. It is a function of a commutation error which is very small for smoothly varying meshes [21].

The filtered Navier-Stokes equations, the generalized SGS turbulent kinetic energy balance equation and the viscous dissipation balance equation are integrated with a sixth order finite difference scheme on a non-uniform staggered grid. A fractional step method is employed.

The computational grid in the physical domain is obtained by mapping a uniform computational grid in the computational domain to the physical domain. Let $D$and Ω be respectively the physical and computational domains, $\overrightarrow{x} = (x_1, x_2, x_3)^T$and $\overrightarrow{\xi} = (\xi_1, \xi_2, \xi_3)^T$ be coordinates in the physical and computational domains, $\overrightarrow{\xi} = \overrightarrow{f}(\overrightarrow{x})$ be a non linear map of physical domain D into the computational domain, and $\Delta_1, \Delta_2, \Delta_3$be uniform grid spacings in the respective directions in computational domain Ω. In this paper we use only uni-directional mappings, $\xi_i = f_i(x_i) \quad (i = 1, ..3)$, and the computational grid in the physical space is constructed as a tensor product of one-directional computational grids. The derivative in the physical space is calculated using the local Jacobian, which can be found numerically using the same stencil and the same order accuracy as finite differencing operator in the computational space. The derivative in computational space (in the one-dimensional case) is approximate as

$$\delta\phi/\delta\xi = (\phi_{i+1} - \phi_{i-1})/2\Delta \tag{49}$$

where Δ is the uniform grid spacing. The derivative in physical space is found as

$$\delta\phi/\delta x = (1/J)\,(\delta\phi/\delta\xi) \tag{50}$$

where J is the Jacobian of the transformation $x \rightarrow \xi$, which can be found numerically by substituting $x$forϕ:

$$J = (\delta x/\delta\xi) = (x_{i+1} - x_{i+1})\,/2\Delta \tag{51}$$

Thus, in this paper, the finite difference in the computational domain with stencil n acting on $\phi$with respect to ξ_1 is

$$(\delta_n\phi/\delta_n\xi_1) \;\equiv\; \frac{\phi(\xi_1 + n\Delta_1/2, \xi_2, \xi_3) - \phi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3)}{n\Delta_1} \tag{52}$$

and the interpolation operator with stencil n acting on ϕ in the $\xi_1$direction is

$$\overline{\phi}^{n\xi_1} = \frac{\phi(\xi_1 + n\Delta_1/2, \xi_2, \xi_3) + \phi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3)}{2} \tag{53}$$

Let *NS6* be the difference between the exact convective term and its discrete approximation. The sixth order accurate scheme for the divergence form of the convective term is given by:

$$\begin{aligned}\frac{\partial u_j u_i}{\partial x_j} - NS6 \equiv\; & \frac{150}{128}\frac{\delta_1}{\delta_1 x_j}\left[\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{1x_j}\right] \\ & - \frac{25}{128}\frac{\delta_3}{\delta_3 x_j}\left[\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{3x_j}\right] \\ & + \frac{3}{128}\frac{\delta_5}{\delta_5 x_j}\left[\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{5x_j}\right]\end{aligned} \tag{54}$$

where the discrete finite difference operator in the physical domain is defined as

$\delta_n\phi/\delta_n x_i = (1/J_i)\,(\delta_n\phi/\delta_n\xi_i)$, where J_i is the local Jacobian of the transformation $x_i \rightarrow \xi_i$ [21].

This numerical scheme has good conservation properties and sixth order accuracy. The sixth-order accurate finite difference approximation of the divergence form of the convective terms conserves mass and momentum ed introduces a weak production of kinetic energy: this production is a function of a commutation error which is very small for smoothly varying meshes [21]; the dynamic procedures for the calculation of the closure coefficients of the production and dissipation terms of the SGS viscous dissipation balance equation (expressed by equations (32), (35), (40) and (46)) are able to compensate dynamically the above mentioned increase of resolved kinetic energy and then allow the large eddy simulation of three-dimensional unsteady flows, also for long time simulations.

5 Results and discussion

Turbulent channel flows (between two flat parallel plates placed at a distance of 2L) are simulated with the proposed Large Eddy Simulation model at different friction-velocity-based Reynolds numbers (Re*), ranging from 395 to 2340. In order to validate the proposed closure relation for the generalized SGS turbulent stress tensor, the numerical results obtained with the proposed model are compared with DNS results (Mansour et al. [9]) and with experimental data (Comte-bellot [2]).

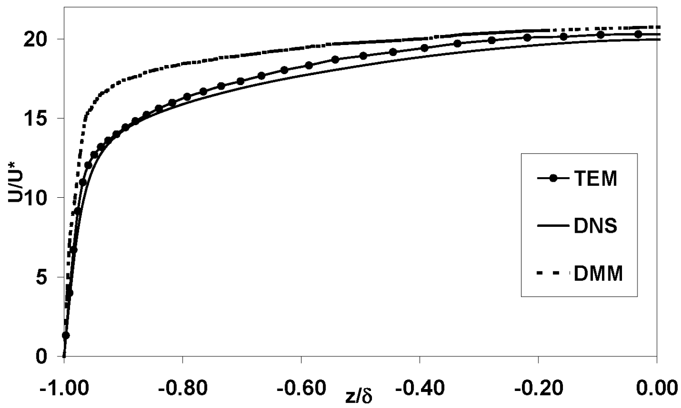

Figure 2: Time-averaged streamwise velocities. Comparison between DNS and LES results obtained with DMM and the proposed model (TEM). Channel flow, Re* = 395.

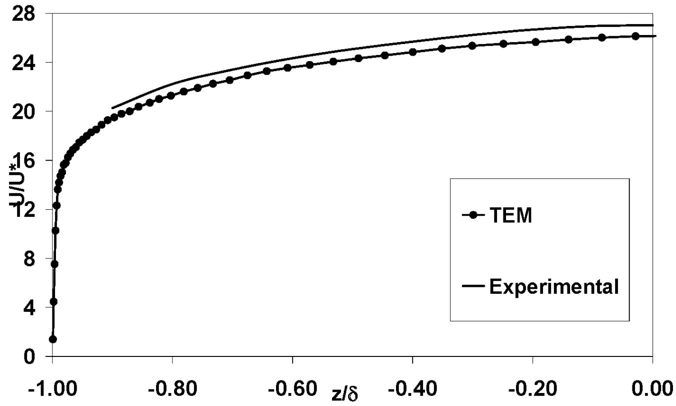

Figure 3: Time-averaged streamwise velocities. Comparison between experimental measurements and LES results obtained with the proposed model (TEM). Channel flow, Re* = 2340.

In Figure 2 is plotted the profile of the time-averaged streamwise velocity component obtained with the proposed model compared with the profile obtained with DNS [9] and the Dynamic Mixed Model, DMM [23], for channel flow at Re* = 395. The figure shows that the profile obtained with the proposed model agrees more with the DNS velocity profile than with the profile obtained with the DMM, both in the boundary layer and in the region inside the channel.

Figure 3 shows the profile of the time-averaged streamwise velocity component for a channel flow at Re*=2340 obtained with the proposed model, compared with the profile of the analogous velocity component measured experimentally [2]. The agreement between the two velocity profiles is very good. Figure 4 compares the profile of the component $\left\{u_1^{\prime}u_3^{\prime}\right\}$ of the Reynolds stress tensor (where indexes (1) and (3) denote, respectively, the streamwise and wall-normal directions), calculated with the proposed model, with the profile of the similar component of the Reynolds stress tensor obtained from experimental measurements [2], for a channel flow at Re* = 2340. Figure 3 shows that at Re* = 2340 the proposed model provides a profile of the component $\left\{u_1^{\prime}u_3^{\prime}\right\}$ in agreement with that of the corresponding component of the Reynolds stress tensor obtained from the experimental measurements.

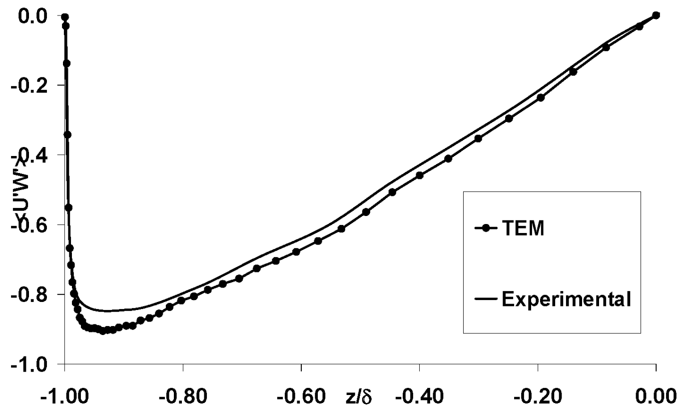

Figure 4: Reynolds stress $\left\{u_1^{\prime}u_3^{\prime}\right\}$. Comparison between experimental measurements and LES results obtained with the proposed model (TEM). Channel flow, Re* = 2340.

Figure 5 shows instantaneous profiles of the terms of the balance equations of E averaged over homogeneous planes, for channel flow at Re*=2340. Figure 5 demonstrates that the balance between production and dissipation of the generalized SGS turbulent kinetic energy is confirmed only in a limited region between the buffer layer and the log layer ($20<z^+<40$) whilst it is not confirmed in other regions of the domain. The viscous dissipation of E is balanced in the viscous sublayer ($z^+<5$) by the viscous diffusion term whilst the production of E is practically negligible. Moving away from the wall, in the

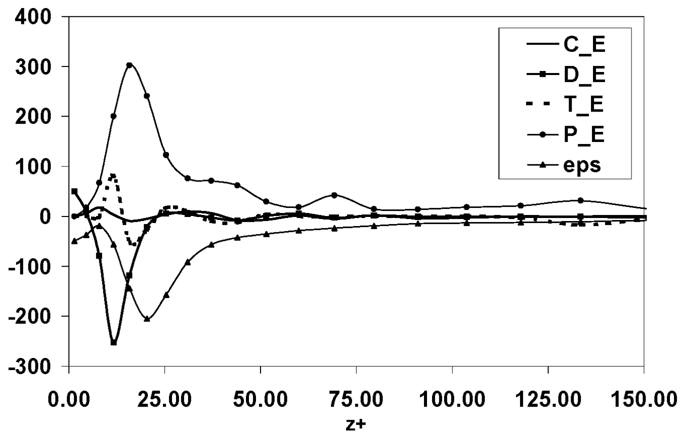

Figure 5: Instantaneous generalized SGS turbulent kinetic energy balance terms averaged over homogeneous planes. Production: P_E; Turbulent transport: T_E; Convection: C_E; Viscous diffusion: D_E; Viscous dissipation: eps. Channel flow, Re*=2340.

first part of the buffer layer, the production term of E increases until it reaches its maximum value ($z^+ \approx 10$) and the terms of turbulent transport and viscous diffusion of E are comparable with the production term of E. In the region between the buffer layer and the log layer ($20 < z^+ < 40$) the convective and turbulent transport terms and the viscous diffusion term are negligible compared with the production and dissipation terms. Only in this limited region there is a balance between the production and the dissipation of E. Towards the centre of the channel ($z^+ > 30$) the viscous dissipation tends towards a minimum but not negligible value. In this region the production term of E is balanced not only by the dissipation but also by the turbulent transport of E.

6 Conclusions

The relations between numerical scheme conservation property of mass, momentum and kinetic energy and the drawbacks of the dynamic Smagorinsky-type turbulence models are shown. A new turbulence model is proposed. The proposed model: a) is able to take into account the anisotropy of the turbulence; b) remove any balance assumption between the production and dissipation of subgrid scale turbulent kinetic energy; c) is able to eliminate the numerical effects produced by the non conservation *a priori* of the resolved kinetic energy. New closure relations for the unknown terms of the subgrid scale viscous dissipation balance equation are proposed. The filtered momentum equations are solved by using a sixth order finite difference scheme. The proposed model is tested for a turbulent channel flow at Reynolds numbers (based on friction velocity and channel half-width) ranging from 395 to 2340.

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