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Tilted Solutions of Einstein's Equations for Bianchi VI_h Cosmologies Expanding with Shear, Acceleration and Rotation

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Abstract. Solutions of Einstein's equations are presented for four representative values of h specifying spatially homogeneous Bianchi VI_h cosmological models using non-comoving coordinates. The perfect fluid in these tilted models has shear, acceleration and rotation with a constant pressure/density ratio. Geometric features of the models are investigated and their kinematic and physical properties calculated.

Keywords: Cosmology, Bianchi VI_h , tilted, non-comoving

1. INTRODUCTION

Historically, in modelling the universe, because of the the relative simplicity of the theoretical treatment, the FRW cosmologies, associated with the names of Friedmann, Robertson and Walker, were first considered. These expanding models were spatially homogeneous and isotropic, the fluid being without shear or rotation, or even acceleration. Later the spatially homogeneous models, not in general isotropic, received attention, being solutions of Einstein's equations that permitted the possibility of expansion with acceleration, shear and rotation of the fluid. The considerable literature on these developments is described for example in the review of exact solutions by Stephani *et al.*[27]. The great majority of these solutions employ comoving coordinates, and assume that at any spacetime point the fluid has a 4-velocity directed normal to the spatially homogeneous 3-space at that point. In the case of such 3-spaces having a simply transitive 3-dimensional isometry, the Bianchi models [2], this means that the fluid may have expansion and shear but neither local acceleration or rotation.

In the simply transitive case it is only when we admit the so called ‘tilted’ models in which the 4-velocity is not normal to the surfaces of symmetry that possible acceleration and rotation can exist. Two cosmological solutions having rotation are listed in [27]. That by Rosquist and Jantzen [25] is an exact solution of Einstein’s equations, while that of Demianski and Grischuk [5] is given only up to a differential equation (and has a non-physical equation of state). As remarked by Stephani *et al.*[27, p.211], if we consider tilted solutions as a whole rather few exact cases are known.

In the methods employed in cosmology using dynamical system theory [31], the exact solutions of Einstein’s equations correspond to equilibrium points of the state space, and it is possible by qualitative analysis to establish general characteristics of the time evolution of the models (see, for example, authors

Hewitt [16], Hewitt and Wainwright [17], Apostolopoulos [1], and very recently Hervik *et al.* [15]). This approach is a valuable one but entirely different from the one made here, which sets out to obtain exact solutions in specific form via the *metric,* the dynamical variables being absorbed into the geometric metric relations (see our equations in section 5).

In this paper, using non-comoving coordinates we present tilted cosmological solutions of Einstein's equations for four representative values of A in Bianchi VI_h geometry, that have expansion, shear, acceleration and rotation. We also examine some aspects of their properties.

For the background to this paper one may refer to several important papers and reviews of relativistic cosmology and the Bianchi models [8-13, 18-20, 23, 26, 27]. However, to establish the notations, definitions and methods to be employed for our purpose we give brief summaries of relevant matter as we proceed.

2. TETRAD DESCRIPTION OF SPACETIME

As a reference basis we set up a tetrad [10, 28, 23] of unit vectors $\{\boldsymbol{e}_a\} = (\boldsymbol{e}_0, \boldsymbol{e}_\alpha)$, where Latin indices run from 0 to 3, Greek from 1 to 3. The vector $\boldsymbol{e}_0$ is timelike so that $\boldsymbol{e}_0 \cdot \boldsymbol{e}_0 = -1$. The tetrad metric coefficients of spacetime are then

$$g_{ab} = \boldsymbol{e}_a . \boldsymbol{e}_b , \tag{2.1}$$

where we take the signature to be +2. The metric will be

$$\mathrm{d}s^2 = g_{ab}\, \boldsymbol{\omega}^a\, \boldsymbol{\omega}^b \tag{2.2}$$

in which the $\boldsymbol{\omega}^a$ are 1-forms dual to the $\boldsymbol{e}_a$. In terms of a coordinate system $\{x^i\} = (t, x, y, z)$ for i = 0,1,2 and 3, respectively we have

$$\boldsymbol{e}_a = e_a{}^i \frac{\partial}{\partial x^i} \quad , \qquad \boldsymbol{\omega}^a = \omega^a{}_i \mathrm{d}x^i \ . \tag{2.3}$$

The commutation functions of the basis are $\gamma^{c}{}_{ab}$ where

$$[\boldsymbol{e}_a . \boldsymbol{e}_b] = \gamma^c_{ab}\,\boldsymbol{e}_c\,, \quad \gamma^{c}{}_{ab} = -\gamma^{c}{}_{ba}\,, \tag{2.4}$$

and these must satisfy the Jacobi identities

$$\boldsymbol{e}_{[a}\gamma^{f}{}_{bc]} - \gamma^{f}{}_{d[a}\gamma^{d}{}_{bc]} = 0\,, \ (\boldsymbol{e}_a\,(F) \equiv e^i{}_a\,\frac{\partial F}{\partial x^i})\,. \tag{2.5}$$

It is also useful to define γ_{abc} by

$$\gamma_{abc} = g_{ad}\gamma^{d}{}_{bc}\ . \tag{2.6}$$

The derivative of $\boldsymbol{e}_a$ in the direction $\boldsymbol{e}_b$ is written

$$\nabla_b\,\boldsymbol{e}_a = \Gamma^c_{ab}\,\boldsymbol{e}_c \tag{2.7}$$

so that the connection coefficients $\Gamma^{\,c}_{ab}$ have the coordinate representation

$$\Gamma^{\,c}_{ab} = \omega^c_j\ e^j_{a\,;\,i}\ e^i_b\,. \tag{2.8}$$

Here the semi-colon denotes the covariant derivative in the x^i system.
From (2.4) and (2.7) we have the relation

$$\gamma^c_{ab} = \Gamma^c_{ba} - \Gamma^c_{ab}\ , \tag{2.9}$$

and for a rigid tetrad frame where the g_{ab} are constants, as will apply effectively in this paper, there is the relation

$$\Gamma_{abc} = \frac{1}{2}\left(\gamma_{cab} - \gamma_{bca} - \gamma_{abc}\right), \tag{2.10}$$

where

$$\Gamma_{abc} = g_{ad}\Gamma^d_{bc}\,. \tag{2.11}$$

The tetrad covariant derivative of a vector **A** is defined as

$$A_{a;\,b} = e^i_b\partial_i A_a - \Gamma^c_{ab}A_c\,, \tag{2.12}$$

$$A^a{}_{;\,b} = e^i_b\partial_i A^a + \Gamma^a_{cb}A^c\,, \tag{2.13}$$

and by extension we have the derivative of a tensor. In particular the metric condition $g_{ab;\,c} = 0$ is imposed leading to, for a rigid frame,

$$\Gamma_{abc} + \Gamma_{bac} = 0, \tag{2.14}$$

consistent with (2.10). Finally the Riemann tensor is given by

$$R^{a}_{bcd} = e_c\left(\Gamma^{a}_{bd}\right) - e_d\left(\Gamma^{a}_{bc}\right) + \Gamma^{e}_{bd}\Gamma^{a}_{ec} - \Gamma^{e}_{bc}\Gamma^{a}_{ed} - \Gamma^{a}_{be}\gamma^{e}_{cd}, \tag{2.15}$$

yielding the Ricci tensor

$$R_{ab} = R^{c}_{acb}. \tag{2.16}$$

3. THE BIANCHI GROUPS

The nine groups of Bianchi [2] are simply transitive isometries G_3 acting on homogeneous spatial hypersurfaces. We take the unit normal ***n*** at any point of each surface to be the *t*-line of our coordinates *t,x,y,z,* the *x,y,z* lines lying in the surface through that point. We label the surfaces as *t* = constant so that we may set

$$n^{a} = \left(1, 0, 0, 0\right), \qquad n_a = \left(-1, 0, 0, 0\right) = -t_{,a} . \tag{3.1}$$

When the orbits of a G_3 group of motions are hypersurfaces then their normals (non null) are geodesic and the surfaces are geodesically parallel [6]. Thus our *t*-line is geodesic and being normal to a hypersurface is without rotation. Therefore any comoving solutions to Einstein's equations that have the fluid 4-velocity in the ***n*** direction are wthout acceleration or rotation of the fluid.

Since the *t*-lines are geodesic normals the spacetime metric will have the coordinate form [29(p. 37)]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \left(i, j = 1, 2, 3\right). \tag{3.2}$$

The isometry G_3 will have three Killing vectors $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$ which satisfy the commutator relations on the spatial 3-legs of a tetrad basis:

$$[\boldsymbol{\xi}_\alpha, \boldsymbol{\xi}_\beta] = C^{\gamma}_{\alpha\beta}\boldsymbol{\xi}_\gamma , \tag{3.3}$$

where the $C^{\gamma}_{\alpha\beta}$, so defined, are the structure constants of the group. These must satisfy the Jacobi identities analogous to (2.5). The Bianchi groups have been categorized by the method of Shucking, Kundt and Behr [13] by expressing the $C^{\lambda}_{\mu\nu}$ in the form

$$C^{\lambda}_{\mu\nu} = \varepsilon_{\mu\nu\alpha} N^{\lambda\alpha} + \delta^{\lambda}_{\nu} A_{\mu} - \delta^{\lambda}_{\mu} A_{\nu}, \tag{3.4}$$

where $\varepsilon_{\mu\nu\alpha}$ is the completely skew unit tensor (ε_{123} =1) and $N^{\mu\nu} = N^{(\mu\nu)}$ while

$$A_{\mu} = \frac{1}{2} C^{\lambda}_{\mu\lambda}. \tag{3.5}$$

The relation

$$N^{\mu\lambda} A_{\lambda} = 0 \tag{3.6}$$

can be deduced from the Jacobi identities. The Bianchi *VI* and *VII* cases have a further invariant *h* given by

$$(1-h) C^{\ \alpha}_{\mu\alpha} C^{\lambda}_{\nu\lambda} + 2h C^{\ \alpha}_{\lambda\mu} C^{\lambda}_{\alpha\nu} = 0. \tag{3.7}$$

For a simply transitive G_3 it is possible to complete a reference tetrad, with $\boldsymbol{e}_0 = n^a$ as in (3.1), by choosing $\boldsymbol{e}_\alpha$ be invariant with respect to the isometry [27 p. 94, 12, 26]. Thus

$$[\boldsymbol{\xi}_\mu . \boldsymbol{e}_\alpha] = 0. \tag{3.8}$$

Since the 3-metric tensor $g_{\alpha\beta}$ of the reference tetrad is given by $\boldsymbol{e}_\alpha . \boldsymbol{e}_\beta$ it follows that $g_{\alpha\beta}$ will also be invariant. Hence the $g_{\alpha\beta}$ will be constant on the surfaces of transitivity and therefore functions only of *t* in spacetime. The metric (3.2) in tetrad form will therefore be

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{\alpha\beta} \boldsymbol{\omega}^{\alpha} \boldsymbol{\omega}^{\beta}, \tag{3.9}$$

the $\boldsymbol{\omega}^{\alpha}$ being duals of the $\boldsymbol{e}_\alpha$ on the spatial triad.

By choosing appropriate members of $\boldsymbol{\xi}_\alpha$ and $\boldsymbol{e}_\beta$ it may be shown from the corresponding Jacobi identities that the $C^\lambda_{\mu\nu}$ are constants in spacetime, and the $\gamma^\lambda_{\mu\nu}$ of the 3-space are functions of t only in spacetime. In a manner similar to (3.4) we may then represent the $\gamma^\lambda_{\mu\nu}$ on the spatial 3-legs of our tetrad, in the form [27 p. 105, 12, 20]:

$$\gamma^\lambda_{\mu\nu} = \varepsilon_{\mu\nu\alpha} n^{\lambda\alpha} + \delta^\lambda_\nu a_\mu - \delta^\lambda_\mu a_\nu \,, \tag{3.10}$$

with $n^{\alpha\beta} = n^{(\alpha\beta)}$ and a_α being functions of t. Again we have the relations

$$a_\mu = \frac{1}{2}\gamma^\lambda_{\mu\lambda}, \tag{3.11}$$

$$n^{\nu\lambda} a_\lambda = 0. \tag{3.12}$$

Finally, it should be noted that by a suitable linear transformation we may arrange that the $\boldsymbol{e}_\alpha$ form with n^a an orthonormal tetrad basis (see section 6).

4. BIANCHI VI_H

In this paper we shall specialise our attention to Bianchi VI_h . We restrict our consideration to the case when

$$N^\alpha_\alpha = 0. \tag{4.1}$$

For this case the 'canonical form' of the generators can be expressed as [27 p. 190]:

$$[\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = 0, \;\; [\boldsymbol{\xi}_3, \boldsymbol{\xi}_1] = (1\text{-}A)\boldsymbol{\xi}_3, \;\; [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] = (1\text{+}A)\boldsymbol{\xi}_2\,, \tag{4.2}$$

where

$$h = -A^2. \tag{4.3}$$

The non-zero structure constants are then (from (3.3)):

$$C_{12}^{2} = -C_{21}^{2} = A+1, \qquad C_{13}^{3} = -C_{31}^{3} = A-1, \tag{4.4}$$

and we see from (3.5) that on the spatial triad:

$$A_{\mu} = \left(A, 0, 0\right). \tag{4.5}$$

In this paper we shall be dealing with the case $A > 0$, which is therefore of class B in the A,B classification of the Bianchi models [12].

The Killing vectors of the isometry, in coordinate contravariant form, are

$$\begin{aligned} \boldsymbol{\xi}_1 &= \{1,\ -(A+1)y,\ (1-A)z\}, \\ \boldsymbol{\xi}_2 &= \{0,\ 1,\ 0\}, \\ \boldsymbol{\xi}_3 &= \{0,\ 0,\ 1\}. \end{aligned} \tag{4.6}$$

From (3.4) we now find for the canonical $N^{\mu\nu}$:

$$N^{\mu\nu} = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}\ . \tag{4.7}$$

The corresponding canonical values of the $\boldsymbol{\omega}^{\alpha}$ which may now be entered into (3.9) [27, p.190] are

$$\begin{aligned} W^1 &= \left(\mathrm{d}x, \quad 0 \quad 0\right) \\ W^2 &= \left(0, \quad \mathrm{e}^{(A+1)x}\mathrm{d}y, \quad 0\right) \\ W^3 &= \left(0, \quad 0, \quad \mathrm{e}^{(A-1)x}\mathrm{d}z\right). \end{aligned} \tag{4.8}$$

Hence (3.2) will have the structure

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a\left(t\right)\mathrm{d}x^2 + b\left(t\right)e^{2(A+1)x}\mathrm{d}y^2 + c\left(t\right)e^{2(A-1)x}\mathrm{d}z^2 + 2r\left(t\right)e^{(A+1)x}\mathrm{d}x\mathrm{d}y. \tag{4.9}$$

Here, for mathematical economy, we omit terms in d*y*d*z* and d*z*d*x*. Moreover, we shall assume a power law dependence on *t*, in the form

$$a(t) = a_1 t^{a_0}, \qquad b(t) = t^{b_0}, \qquad c(t) = t^{c_0}, \qquad r(t) = r_1 t^{r_0}, \qquad (4.10)$$

where $a_{0,} b_0, c_0, r_0, a_1$ and r_1 are constants.

5. SPACETIME IN THE PRESENCE OF A PERFECT FLUID

We denote by p the pressure and μ the density of a perfect fluid. Let its tetrad (assumed orthonormal with a view to section 6) unit 4-velocity be u^a. In this paper we shall not assume comoving coordinates but will make the simplification $u^3 = 0$. Thus

$$u^a = \left(u^0, u^1, u^2, 0\right), \qquad (5.1)$$

where

$$\left(u^0\right)^2 = 1 + \left(u^1\right)^2 + \left(u^2\right)^2. \qquad (5.2)$$

The energy tensor T_b^a will have the form

$$T_b^a = \left(\mu + p\right) u^a u_b + \delta_b^a p = G_b^a, \qquad (5.3)$$

where G_b^a is the Einstein tensor, having chosen geometrized units $\left(8\pi G = c = 1\right)$.

We have therefore the necessary relations relative to the tetrad:

$$G_3^0 = G_3^1 = G_3^2 = G_0^3 = G_1^3 = G_2^3 = 0,$$

$$G_1^0 + G_0^1 = 0,$$

$$G_1^2 - G_2^1 = 0,$$

$$G_2^0 + G_0^2 = 0. \qquad (5.4)$$

$$X_1 \equiv \left(G_0^0 - G_3^3\right)\left(G_1^1 - G_3^3\right) - G_1^0 G_0^1 = 0,$$

$$X_2 \equiv \left(G_0^0 - G_3^3\right)\left(G_2^2 - G_3^3\right) - G_2^0 G_0^2 = 0,$$

$$X_3 \equiv \left(G_1^1 - G_3^3\right)\left(G_2^2 - G_3^3\right) - G_2^1 G_1^2 = 0. \tag{5.5}$$

$$p = G_3^3,$$

$$\mu = 2G_3^3 - G_0^0 - G_1^1 - G_2^2. \tag{5.6}$$

$$\left(u^0\right)^2 = \left(G_3^3 - G_0^0\right)/D,$$

$$\left(u^1\right)^2 = \left(G_1^1 - G_3^3\right)/D,$$

$$\left(u^2\right)^2 = \left(G_2^2 - G_3^3\right)/D, \tag{5.7}$$

where

$$D = 3G_3^3 - G_0^0 - G_1^1 - G_2^2. \tag{5.8}$$

These results will be employed in seeking tilted non-comoving solutions to Einstein's equations subject to a Bianchi VI_h group of isometries.

6. Solutions

Substitution into the relations (5.5) reveals that for a solution to Einstein's equations we require

$$a_0 = 2, \qquad r_0 = 1 + b_0/2. \tag{6.1}$$

Accordingly, we rewrite (4.9) in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + k^2\left(1+m^2\right)t^2\mathrm{d}x^2 + t^{2(-q+s)}e^{2(A+1)x}\mathrm{d}y^2 + t^{2(q+s)}e^{2(A-1)x}\mathrm{d}z^2 + 2mkt^{(1-q+s)}e^{(A+1)x}\mathrm{d}x\mathrm{d}y, \tag{6.2}$$

k, m, q and s being constants.

An orthonormal tetrad to describe (6.2) relative to the x^i system is:

$$\boldsymbol{\omega}^0 = \{1, 0, 0, 0\},$$

$$\boldsymbol{\omega}^1 = \left\{0,\ k\left(1+m^2\right)^{1/2} t,\ m/\left(1+m^2\right)^{1/2} t^{(-q+s)}e^{(A+1)x},\ 0\right\},$$

$$\boldsymbol{\omega}^2 = \left\{0,\ 0,\ t^{(s-q)}/\left(1+m^2\right)^{1/2} e^{(A+1)x},\ 0\right\},$$

$$\boldsymbol{\omega}^3 = \left\{0,\ 0,\ 0,\ t^{(q+s)} e^{(A-1)x}\right\}. \tag{6.3}$$

The corresponding (dual) system of the $\boldsymbol{e}_a$ is:

$$\boldsymbol{e}_0 = \{1,\ 0,\ 0,\ 0\},$$

$$\boldsymbol{e}_1 = \left\{0,\ 1/\left[k\left(1+m^2\right)^{1/2} t\right],\ 0,\ 0\right\},$$

$$\boldsymbol{e}_2 = \left\{0,\ -m/\left[k\left(1+m^2\right)^{1/2} t\right],\ t^{(q-s)}\left(1+m^2\right)^{1/2} e^{-(A+1)x},\ 0\right\},$$

$$\boldsymbol{e}_3 = \left\{0,\ 0,\ 0,\ t^{-(q+s)} e^{(1-A)x}\right\}, \tag{6.4}$$

and for an orthonormal tetrad we verify that $\boldsymbol{e}_0 . \boldsymbol{e}_\alpha = 0$, $\boldsymbol{e}_\alpha . \boldsymbol{e}_\beta = \delta_{\alpha\beta}$. It may be confirmed that the spatial parts of the $\boldsymbol{e}_\alpha$ in (6.4) satisfy the invariant conditions (3.8), the $\boldsymbol{\xi}_\alpha$ being given by (4.6).

The non-zero values of the $\gamma^\lambda_{\mu\nu}(t)$ are now found to be

$$\gamma^1_{12} = -\gamma^1_{21} = -m(A+1))/E,$$

$$\gamma^2_{12} = -\gamma^2_{21} = -(A+1)/E,$$

$$\gamma^3_{23} = -\gamma^3_{32} = m(A-1)/E,$$

$$\gamma^3_{31} = -\gamma^3_{13} = (A-1)/E, \tag{6.5}$$

where

$$E = k\left(1+m^2\right)^{1/2} t, \tag{6 .6}$$

and these satisfy the 3-space Jacobi identities.

We now derive from (3.10) and (3.11) the tetrad values

$$a_\mu = \{-A/E,\ Am/E,\ 0\}, \tag{6.7}$$

$$n^{\mu\nu} = \begin{matrix} 0 & 0 & -m/E \\ 0 & 0 & -1/E \\ -m/E & -1/E & 0 \end{matrix}, \qquad (6.8)$$

so that we can confirm the results

$$n^{\alpha}_{\alpha} = 0, \qquad n^{\alpha\beta} a_{\beta} = 0. \qquad (6.9)$$

The ensuing analysis makes it convenient to write

$$k^2 = k_0, \qquad m^2 = m_0. \qquad (6.10)$$

Calculation of X_3 in (5.5) produces an expression quadratic in both k_0 and m_0, unless $A = 0$ in which case the presence of k_0 is linear. $A = 0$ was the case considered by Rosquist and Jantsen [25]. When $A > 0$ computation in the general case becomes rather intractable. Here we shall employ a simplifying device of choosing k_0 so that the term in the m_0 quadratic which is independent of m_0 vanishes. We can then obtain for k_0 the value

$$k_0 = -(A+1)/\{s(q+s-1)\}. \qquad (6.11)$$

Solving for m_0 by setting $X_3 = 0$ then produces:

$$m_0 = \{ \left(-q^2 - 2q + 6qs - s^2 - 2s - 1\right) A - q^2 - 2q - 2qs - 9s^2 + 6s - 1 \}/\left\{(q - s + 1)^2 (A+1)\right\}, \qquad (6.12)$$

or

$$m_0 = 0. \qquad (6.13)$$

Adopting (6.11) and (6.12) we have to solve $X_1 = X_2 = 0$. The result is two simultaneous equations for q and s with A as a parameter. The general case is still difficult, but fixing a value for A we can look for real values for q and s to satisfy the Einstein equations.

We illustrate the procedure by choosing

$$A = 1/3, \qquad (h = -1/9). \tag{6.14}$$

Substitution in $X_1 = X_2 = 0$ yields *first* the solution

$$q = -1/4,\ s = 1/4 \Rightarrow k_0 = 16/3,\ \ m_0 = 0. \tag{6.15}$$

Then (5.6) and (5.7) produce

$$u^a = (1,0,0,0), \quad p = 1/(12t^2), \quad \mu = 1/(4t^2), \tag{6.16}$$

so that for this simple solution the coordinates are *comoving* and the fluid represents pure radiation. The metric is

$$ds^2 = -dt^2 + (16/3)t^2 dx^2 + e^{8x/3} t dy^2 + e^{-4x/3} dz^2. \tag{6.17}$$

This result is to be found by setting $A = 1/3$ in the comoving perfect fluid Bianchi VI_h solution obtained by Collins [4]. Being non-tilted it is acceleration and rotation free.

In fact we can reproduce Collins' solution for all $A \geq 0$ by adopting (6.13), so that there is no $dxdy$ term in the metric, and then substituting in X_1, X_2 the relations

$$q = A(2-3\gamma)/(2\gamma), \quad s = (2-\gamma)/(2\gamma), \tag{6.18}$$

with, from (6.11),

$$k_0 = 4\gamma^2 / \{(2-\gamma)(3\gamma-2)\}. \tag{6.19}$$

We then have Collins'solution with $p = (\gamma - 1)\mu$. Unfortunately, the simple relation between s and γ in (6.18) features only in A class models (or untilted B models), as in the references [4] and [25].

Reverting to the case (6.12), with $m_0 \neq 0,$ the above substitution procedure also furnishes for q and s the equations

$$s = \left(7680q^3 - 7024q^2 - 2461q + 693\right)/3177, \tag{6.20}$$

$$768q^4 - 1336q^3 + 524q^2 + 42q - 81 = 0. \tag{6.21}$$

We can solve (6.21) to arbitrarily high precision, but for the sake of compactness, without rounding up or down we give results to 6 decimal places. There is a solution

$$q = -0.305694, \quad s = 0.179268, \quad k_0 = 6.602869, \quad m_0 = 0.037896, \tag{6.22}$$

which leads to

$$p = 0.031541/t^2, \quad \mu = 0.075506/t^2, \quad p/\mu = 0.417732, \tag{6.23}$$

which suggests a fluid somewhat stiffer than pure radiation. From (5.7) we derive the fluid 4-velocity:

$$u^a = \left(1.077661,\ 0.148065,\ \text{-}0.373403,\ 0\right). \tag{6.24}$$

The metric will be

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 6.853096t^2\mathrm{d}x^2 + t^{0.969924}e^{8x/3}\mathrm{d}y^2 + t^{-0.252851}e^{-4x/3}\mathrm{d}z^2 + 1.000455t^{1.484962}e^{4x/3}\mathrm{d}x\mathrm{d}y. \tag{6.25}$$

The shear and spin of the fluid will be discussed in section 7.

We have thus an exact cosmological solution of Einstein's equations (that is, obtainable from an algebraic equation solvable to arbitrarily high precision), with expansion, shear, acceleration and rotation and a $p/\mu =$ constant equation of state.

If we set $A = 1/5$ and proceed similarly we find the equations

$$s = \left(115248q^3 - 118328q^2 - 30425q + 10083\right)/49551, \tag{6.26}$$

$$57624q^4 - 100324q^3 + 38689q^2 - 1576q - 4326 = 0, \tag{6.27}$$

yielding a solution

$$q = -0.243434, \quad s = 0.177893, \quad k_0 = 6.330701, \quad m_0 = 0.131478, \quad (6.28)$$

$$p = 0.038939/t^2, \quad \mu = 0.093396/t^2, \quad p/\mu = 0.416924. \quad (6.29)$$

$$u^a = (1.163633,\ 0.235743,\ -0.546322,\ 0). \quad (6.30)$$

The results describing solutions for $A = 1/3,\ 1/5,\ 1/7$ and $1/9$ are given in Table 1. At the precision level to 6 decimal places the Einstein function $G_b^a - T_b^a$ (see (5.3)), is of order $10^{-6}/t^2$ for a,b = 0 to 3. Raising the precision level to 12 decimal places, it is of order $10^{-12}/t^2$, and so on.

7. SPIN, SHEAR AND ACCELERATION IN TILTED MODELS

In tilted models, where the fluid 4-velocity **u** is not aligned with **n** the normal to the surfaces of symmetry $S(t)$, the angle of tilt is defined as β given by

$$\cosh(\beta) = -u^a n_a, \quad \beta(t) > 0. \quad (7.1)$$

In their paper on tilted homogeneous models, King and Ellis [18] have defined $\tilde{c}^a$ as the direction of the projection of u^a on the surface $S(t)$ by

$$\tilde{h}_b^a u^b = \sinh\beta\ \tilde{c}^a \ \ (\tilde{c}^a \tilde{c}_a = 1,\ \tilde{c}^a n_a = 0), \quad (7.2)$$

$$\tilde{h}_{ab} = g_{ab} + n_a n_b. \quad (7.3)$$

They also define c^a as the direction of the component of n^a perpendicular to u^a, by the relation

$$h_b^a n^b = -\sinh\beta\ c^a \quad (c^a c_a = 1,\ c^a u_a = 0) \quad (7.4)$$

$$h_{ab} = g_{ab} + u_a u_b. \quad (7.5)$$

One can then show (cf. [18]) that for a perfect fluid

$$\dot{u}^a = \tanh\beta \; \mathrm{d}p/\mathrm{d}\mu \; \theta \; c^a \, , \tag{7.6}$$

$$\omega^a = \frac{1}{2}\sinh\beta \; \eta^{abcd} u_b \tilde{c}_{c\,;\,d} \, , \tag{7.7}$$

where η^{abcd} $\left(\eta^{0123} = 1\right)$ is the completely skew unit 4-tensor , θ $\left(= u^a{}_{;a}\right)$ is the expansion , $\dot{u}^a$ $\left(= u^a{}_{;b}\, u^b\right)$ the acceleration and ω^a $\left(= \frac{1}{2}\eta^{abcd} u_b u_{c;\,d}\right)$ the rotation vector of the fluid. Thus we see that *in a tilted model there will be in general both acceleration and rotation of the fluid.* Moreover it can be shown that (cf.[18]):

$$u_{a\,;\,b} = \cosh\beta \; n_{a\,;\,b} + \sinh\beta \; \tilde{c}_{a\,;\,b} - \dot{\beta} c_a n_b \, , \tag{7.8}$$

and since the fluid shear is given by

$$\sigma_{ab} = \frac{1}{2}\left(u_{a\,;\,\mathrm{b}} + u_{b\,;\,a}\right) + \frac{1}{2}\left(\dot{u}_a u_b + \dot{u}_b u_a\right) - \frac{1}{3}\theta h_{ab} \, , \tag{7.9}$$

we can expect a tilted model to have shear. In fact King and Ellis [18] prove that there can be no tilted models with vanishing shear of a perfect fluid.

In our case , with n^a given by (3.1) and u^a as in (5.1), it follows that relative to our tetrad:

$$\begin{aligned} &\cosh\beta = u^0 , \\ &\tilde{c}^a = \left(0,\;\; u^1/\sinh\beta,\;\; u^2/\sinh\beta,\;\; 0\right), \\ &c^a = \left(\sinh\beta,\;\; u^1\coth\beta,\;\; u^2\coth\beta,\;\; 0\right). \end{aligned} \tag{7.10}$$

In Table 1 the results are shown for our models in respect of $\sigma = (\sigma_{ab}\sigma^{ab}/2)^{1/2}$, $\omega = \left(\omega^a\omega_a\right)^{1/2}$, $\dot{u} = \left(\dot{u}^a\dot{u}_a\right)^{1/2}$ and θ , and these can be verified to be in agreement with the formulae given here.

Table 1: Metric parameters and kinematic/physical quantities, as defined in the text, for four representative values of A for which spatially homogeneous, tilted solutions of Einstein's equations have been obtained.

A	$1/3$	1/5	1/7	1/9
q	-0.305694	-0.243434	-0.217387	-0.202756
s	0.179268	0.177893	0.180210	0.182251
k_0	6.602869	6.330701	6.114454	5.974083
m_0	0.037896	0.131478	0.187222	0.222323
μ	$0.075506/t^2$	$0.093396/t^2$	$0.091847/t^2$	$0.088582/t^2$
p/μ	0.417732	0.416924	0.419630	0.422221
$\dot{u}$	$0.236713/t$	$0.350161/t$	$0.411737/t$	$0.453252/t$
θ	$1.520263/t$	$1.642477/t$	$1.716832/t$	$1.769161/t$
σ	$0.657996/t$	$0.721521/t$	$0.764178/t$	$0.794868/t$
ω	$0.081427/t$	$0.088784/t$	$0.089857/t$	$0.090158/t$
β	0.391602	0.564545	0.649768	0.703811

8. Geometrical features

Invariants

Taking the representative value $A = 1/5$ we find

$$\begin{aligned} R_{abcd}R^{abcd} &= -1.024109/t^4, \\ R_{ab}R^{ab} &= 0.013271/t^4, \\ R &= -0.023422/t^2. \end{aligned} \tag{8.1}$$

Combined with (6.29) we see that these results indicate a physical singularity at $t = 0$, the rest of spacetime being free of singularity.

Petrov type

Adopting an appropriate complex null tetrad for the metrics we find that these solutions are of Petrov type I. Thus for $A = 1/5$, in the notation of Stephani *et al.* [27 (p. 54)], calculation shows that

$$\mathrm{I} = -0.065654/t^4, \qquad \mathrm{J} = 0.002159/t^6, \tag{8.2}$$

so that the discriminant

$$\mathrm{I}^3 - 27\mathrm{J}^2 = -0.000408/t^{12} \neq 0. \tag{8.3}$$

Since I and J are both real with I < 0, J > 0 and the discriminant is < 0, the metrics are, in the notation of McIntosh and Arianrhod [21], of Petrov type I (M^-) and so the four principal null directions associated with the Weyl tensor are linearly independent.

The Weyl electric/magnetic decomposition

For $A = 1/5$ calculation gives for the invariants:

$$\begin{aligned} E_{ab}E^{ab} &= 0.520295/t^4, \\ H_{ab}H^{ab} &= 0.651603/t^4. \end{aligned} \tag{8.4}$$

We note that these quantities vary as t^{-4} in contrast with the mass density $\propto t^{-2}$, so that the energy of gravitational waves may be an important factor in the regime $t \to 0$. The Weyl tensor is neither purely electric nor magnetic and, consistent with the fact that I in (8.2) is real [22, 3], we find for the invariants:

$$\begin{aligned} E_{ab}H^{ab} &= 0, \\ E_{ab}E^{ab} - H_{ab}H^{ab} &= 2\mathrm{I}. \end{aligned} \tag{8.5}$$

Here there is an analogy to the electromagnetic case $E.H = 0$ when E and H are directed at right angles.

Extrinsic curvature of the surfaces of transitivity

The extrinsic curvature, in our geodesic normal x^i system, is [24]:

$$K_{ij} = -n_{ij} = -\left\{ {0 \atop ij} \right\} = -\frac{1}{2}\partial g_{ij} / \partial t \ \ (i,j = 1,2,3). \tag{8.6}$$

Hence the invariant:

$$K = K_i^i = -\frac{1}{2} g^{ij} \partial g_{ij} / \partial t, \tag{8.7}$$

which gives for our models

$$K = -(2s+1)/t. \tag{8.8}$$

Thus the extrinsic scalar is < 0, infinite at $t = 0$ and vanishing as $t \to \infty$.

Riemannian curvatures of 2-surfaces on $S(t)$

Our interest is in the 2-spaces formed by the vectors $\hat{x} = (1,0,0)$, $\hat{y} = (0,1,0)$ and $\hat{z} = (0,0,1)$ on the surfaces of transitivity. For the 2-space $(\hat{y}, \hat{z})$ one obtains the Riemannian curvature [29 p. 17] :

$$K_1 = \left(1 - A^2\right) / \left(k_0 t^2\right). \tag{8.9}$$

For $(\hat{z}, \hat{x})$:

$$K_2 = \frac{m_0\left(1 - A^2\right) - (1 - A)^2}{k_0\left(m_0 + 1\right) t^2}. \tag{8.10}$$

For $(\hat{x}, \hat{y})$:

$$K_3 = -\left(A + 1\right)^2 / \left(k_0 t^2\right). \tag{8.11}$$

Thus for $A = 1/5$:

$$K_1 = 0.151641/t^2,$$
$$K_2 = -0.071726/t^2,$$
$$K_3 = -0.227462/t^2.$$
$$\tag{8.12}$$

The 3-spaces $t = \text{constant}$ are thus not isotropic (unlike the VII_0 model considered by Demianski and Grischuk [5], which has a flat 3-space). We see that negative Riemannian curvature is dominant at any point (x, y, z). Anisotropy continues, becoming infinite as $t \to 0,$ while as $t \to \infty$ it persists as the curvatures tend to zero.

The scalar curvature R^*

In view of (8.12) the 3-curvature of the surfaces $S(t)$ may be expected to be negative. It will be of incidental interest to employ the results of (6.7) and (6.8) from which R^* may be calculated [19]. Thus in our tetrad basis

$$R^* = \left(n_\alpha^\alpha\right)^2/2 - 6a_\alpha a^\alpha - n^{\alpha\beta}n_{\alpha\beta,} \tag{8.13}$$

yielding

$$R^* = -2\left(3A^2 + 1\right)/\left(k_0 t^2\right). \tag{8.14}$$

We may verify this result by making use of the extrinsic curvature, or second fundamental form, (8.6). That is, if we write $\dot{K} = \partial K/\partial t$ then, in coordinate form, from a result given by Eisenhart [7 p.149], we may deduce that

$$R^* = 2\dot{K} - K^2 - K_{ij}K^{ij} + R, \tag{8.15}$$

where R is the scalar curvature of spacetime. Calculation from the metric (6.2) confirms (8.14). As expected, the scalar 3-curvature is negative, infinite at $t = 0$ and vanishing as $t \to \infty$.

It is to be noted that the curvatures K_i (i = 1.2.3) and R^* are of the same order as the density μ, both as $t \to 0$ and as $t \to \infty$. Thus both influences are present at all epochs.

Homotheity (*similarity*)

The models may be shown to possess a homothetic vector ξ_i satisfying

$$\xi_{i;j} + \xi_{j;i} - 2g_{ij} = 0. \tag{8.16}$$

Here *i,j* run from 0 to 3. In coordinate contravariant form the vector is

$$\xi^i = t\partial_t + (1+q-s)\, y\partial_y + (1-q-s)\, z\partial_z. \tag{8.17}$$

The spacetime of the models is therefore self-similar. As Wainwright has shown [30], this implies that an equation of state $p = (\gamma - 1)\mu$ $(\gamma = \text{constant})$ obtains. This is in accordance with our results in Table I.

11. CONCLUSION

BianchiVI_h solutions to Einstein's equations have been obtained for several values of *h*, or *A*, in non-comoving coordinates. Aspects of the geometrical and physical properties of these tilted models have been examined. The models are an example of alternative dynamics for the universe whose relevance must of course be decided by established observational data. However, it also provides an illumination of some geometrical and dynamical facets of Einstein's equations.

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