

Expansion of the Universe from a 5D Vacuum within a Nonperturbative Scalar Field Formalism

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Abstract

We study the dynamics of the quantum scalar field responsible for inflation in different epochs of the evolution of the universe by using a recently introduced nonperturbative formalism from a 5D apparent vacuum.

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1 Introduction and basic formalism

Multi-dimensional theories of gravity and, in particular, the recent Braneworld cosmological scenario, in which the universe of our perception is a submanifold embedded in a high dimensional manifold, have generated very much interest in

mathematical and physical aspects [1, 2, 3] of embedding theories based on four dimensional general relativity (4DGR). According to 4DGR, the model that describes the large scale structure of the universe is a 4D manifold equipped with a pseudo-Riemannian metric structure. The possibility of the existence of more than 4D was taken into account not very long after the final formulation of 4DGR by Kaluza and Klein (KK) [4]. In the original theory, the extra dimension was considered as topologically compact and the 4D spacetime was assumed independent of the fifth dimension. In this theory the universe was considered as a hyper-cylinder, locally homeomorphic to $\mathfrak{R}^4 \times S^1$. In the modern version of the KK theory, these assumptions are removed and the result is the unification of gravitation and electromagnetism with induced sources (mass and charge) [5, 6, 7]. Another recent cosmological scenario claims that the universe of our perception is a brane of 4D embedded in a n D bulk of at least 5D [8]. The main ansatz is that matter is confined on the brane and only gravity and electromagnetic fields can propagate on the bulk. Mathematically, these theories are based on a basic theorem of Riemannian geometry due to Campbell, who stated it for the first time [9], and Magaard [10], who presented a strict proof. This is a local embedding theorem where every smooth analytic n D manifold can be locally and isometrically embedded into some other $(n + 1)$ D one, which is Ricci-flat, $R_{AB} = 0$ (In our conventions, capital Latin indices A, B, \dots run from 0 to 4, whereas greek indices μ, ν, \dots from 0 to 3). In a sense, this is equivalent to the local embedding of Einstein 4DGR, into some 5D relativity in vacuum where matter does not exist. The idea of this approach is to explain matter in 4D as a manifestation of pure geometry in higher dimensions. In a cosmological framework, scalar fields have been recognized to be responsible for the expansion of the universe [11], and have been proposed to explain inflation [12], as well as the present accelerated quintessential expansion [13]. The main goal of this paper is the study of a power-law expansion of the universe from a 5D apparent vacuum, defined by a 5D flat metric and a purely kinetic Lagrangian for a scalar field minimally coupled to gravity. For this purpose, we consider the 5D metric introduced in [14, 15],

$$dS^2 = \epsilon \left(\psi^2 dN^2 - \psi^2 e^{2N} dr^2 - d\psi^2 \right), \quad (1)$$

where $dr^2 = dx^2 + dy^2 + dz^2$. The coordinates (N, r) are dimensionless, the fifth coordinate ψ has spatial units and ϵ is a dimensionless parameter that can take the values $\epsilon = \pm 1$, accounting for the two possible signatures of the metric.

The metric (1) describes a flat 5D manifold ($R_{BCD}^A = 0$) in apparent vacuum ($G_{AB} = 0$). We consider a diagonal metric because we are dealing only with gravitational effects, which are the important ones in the global evolution for the universe. To describe neutral matter in a 5D geometrical vacuum (1)

we can consider the Lagrangian

$${}^{(5)}L(\varphi, \varphi_{,A}) = -\sqrt{\left|\frac{{}^{(5)}g}{{}^{(5)}g_0}\right|} {}^{(5)}\mathcal{L}(\varphi, \varphi_{,A}), \quad (2)$$

where $|{}^{(5)}g| = \psi^8 e^{6N}$ is the absolute value of the determinant for the 5D metric tensor with components g_{AB} given by (1) and $|{}^{(5)}g_0| = \psi_0^8 e^{6N_0}$ is a constant of dimensionalization determined by $|{}^{(5)}g|$ evaluated at $\psi = \psi_0$ and $N = N_0$. In this work we consider $N_0 = 0$, so that ${}^{(5)}g_0 = \psi_0^8$. Here, the index “0” denotes the values at the end of inflation (i.e., when the scale factor of the universe satisfies $\ddot{b} = 0$). Furthermore, we consider an action

$$I = -\int d^4x d\psi \sqrt{\left|\frac{{}^{(5)}g}{{}^{(5)}g_0}\right|} \left[\frac{{}^{(5)}R}{16\pi G} + {}^{(5)}\mathcal{L}(\varphi, \varphi_{,A}) \right], \quad (3)$$

for a scalar field φ , which is minimally coupled to gravity. Here, ${}^{(5)}R$ is the 5D Ricci scalar, which of course, is zero for the 5D flat metric (1) and G is the gravitational constant, which we consider independent of the extra dimension. A different treatment was considered, for example, in [3]. Since the 5D metric (1) describes a manifold in apparent vacuum, the Lagrangian density \mathcal{L} in (2) must be only kinetic in origin,

$${}^{(5)}\mathcal{L}(\varphi, \varphi_{,A}) = \frac{1}{2} g^{AB} \varphi_{,A} \varphi_{,B}. \quad (4)$$

Considering the metric (1) and the Lagrangian (2), we obtain the equation of motion for φ ,

$$\left(2\psi \frac{\partial\psi}{\partial N} + 3\psi^2\right) \frac{\partial\varphi}{\partial N} + \psi^2 \frac{\partial^2\varphi}{\partial N^2} - \psi^2 e^{-2N} \nabla_r^2 \varphi - 4\psi^3 \frac{\partial\varphi}{\partial\psi} - 3\psi^4 \frac{\partial N}{\partial\psi} \frac{\partial\varphi}{\partial\psi} - \psi^4 \frac{\partial^2\varphi}{\partial\psi^2} = 0, \quad (5)$$

where $\partial N/\partial\psi$ and $\partial\psi/\partial N$ are zero because the coordinates (N, \vec{r}, ψ) are independent.

1.1 5D normalization

Equation (5) can be written as

$$\overset{\star\star}{\varphi} + 3 \overset{\star}{\varphi} - e^{-2N} \nabla_r^2 \varphi - \left[4\psi \frac{\partial\varphi}{\partial\psi} + \psi^2 \frac{\partial^2\varphi}{\partial\psi^2} \right] = 0, \quad (6)$$

where the over-star denotes derivative with respect to N and $\varphi \equiv \varphi(N, \vec{r}, \psi)$. Now we proceed with the quantization procedure for the scalar field $\varphi(N, \vec{r}, \psi)$

in the standard way. We start establishing the commutator relation between φ and $\Pi^N = \partial\mathcal{L}/\partial\varphi_{,N} = g^{NN}\varphi_{,N}$ that results

$$\left[\varphi(N, \vec{r}, \psi), \Pi^N(N, \vec{r}', \psi')\right] = ig^{NN}\delta^{(3)}(\vec{r} - \vec{r}')\delta(\psi - \psi'), \quad (7)$$

being $g^{NN} = \psi^{-2}$. In order to simplify the structure of (6), we find convenient to express φ as $\varphi = \chi e^{-3N/2}(\psi_0/\psi)^2$. Thus, equation (6) yields

$$\overset{\star\star}{\chi} - \left[e^{-2N}\nabla_r^2 + \left(\psi^2 \frac{\partial^2}{\partial\psi^2} + \frac{1}{4} \right) \right] \chi = 0, \quad (8)$$

which is a 5D generalized Klein-Gordon like equation for the field $\chi(N, \vec{r}, \psi)$. The field $\chi(N, \vec{r}, \psi)$ can be Fourier expanded as

$$\chi(N, \vec{r}, \psi) = \quad (9)$$

$$\frac{1}{(2\pi)^{3/2}} \int d^3k_r \int dk_\psi \left[a_{k_r k_\psi} e^{i(\vec{k}_r \cdot \vec{r} + k_\psi \psi)} \xi_{k_r k_\psi}(N, \psi) + a_{k_r k_\psi}^\dagger e^{-i(\vec{k}_r \cdot \vec{r} + k_\psi \psi)} \xi_{k_r k_\psi}^*(N, \psi) \right],$$

where the asterisk denotes complex conjugation and $(a_{k_r k_\psi}, a_{k_r k_\psi}^\dagger)$ are the annihilation and creation operators which satisfy the algebra

$$\left[a_{k_r k_\psi}, a_{k'_r k'_\psi}^\dagger \right] = \delta^{(3)}(\vec{k}_r - \vec{k}'_r) \delta(k_\psi - k'_\psi), \quad (10)$$

$$\left[a_{k_r k_\psi}^\dagger, a_{k'_r k'_\psi}^\dagger \right] = \left[a_{k_r k_\psi}, a_{k'_r k'_\psi} \right] = 0, \quad (11)$$

and the following commutation relation between χ and $\overset{\star}{\chi}$ must be fulfilled

$$\left[\chi(N, \vec{r}, \psi), \overset{\star}{\chi}(N, \vec{r}', \psi') \right] = i\delta^{(3)}(\vec{r} - \vec{r}')\delta(\psi - \psi'). \quad (12)$$

The commutation relation (12) can be written in terms of the scalar modes $\xi_{k_r k_\psi}(N, \psi)$ in the form

$$\xi_{k_r k_\psi} \left(\overset{\star}{\xi}_{k_r k_\psi} \right)^* - \left(\xi_{k_r k_\psi} \right)^* \overset{\star}{\xi}_{k_r k_\psi} = i, \quad (13)$$

so that this normalization condition is completely equivalent to (12). Therefore, a solution for the scalar modes $\xi_{k_r k_\psi}(N, \psi)$ that satisfies (13) corresponds automatically to a solution for $\chi(N, \vec{r}, \psi)$ that satisfies (12). By inserting (9) into (8) we obtain that the dynamical equation for the modes $\xi_{k_r k_\psi}(N, \psi)$ is given by

$$\overset{\star\star}{\xi}_{k_r k_\psi} + k_r^2 e^{-2N} \xi_{k_r k_\psi} + \psi^2 \left(k_\psi^2 - 2ik_\psi \frac{\partial}{\partial\psi} - \frac{\partial^2}{\partial\psi^2} - \frac{1}{4\psi^2} \right) \xi_{k_r k_\psi} = 0. \quad (14)$$

To solve this equation we propose the ansatz

$$\xi_{k_r k_\psi}(N, \psi) = \xi_{k_r}^{(1)}(N) \xi_{k_\psi}^{(2)}(\psi), \tag{15}$$

such that (14) can be written as two differential equations

$$\xi_{k_r}^{** (1)} + [k_r^2 e^{-2N} - \alpha] \xi_{k_r}^{(1)} = 0, \tag{16}$$

$$\frac{d^2 \xi_{k_\psi}^{(2)}}{d\psi^2} + 2ik_\psi \frac{d\xi_{k_\psi}^{(2)}}{d\psi} - \left(k_\psi^2 - \frac{(1/4 - \alpha)}{\psi^2} \right) \xi_{k_\psi}^{(2)} = 0, \tag{17}$$

with $\alpha = 2k_\psi^2 \psi^2$. Solving (16) and (17) we obtain

$$\xi_{k_r}^{(1)}[x(N)] = A_1 \mathcal{H}_\nu^{(1)}[x(N)] + A_2 \mathcal{H}_\nu^{(2)}[x(N)], \tag{18}$$

$$\xi_{k_\psi}^{(2)} = e^{-ik_\psi \vec{\psi}} \left[B_1 \psi^{(\frac{1}{2} + \sqrt{\alpha})} + B_2 \psi^{(\frac{1}{2} - \sqrt{\alpha})} \right], \tag{19}$$

where $\mathcal{H}_\nu^{(1,2)}[x(N)] = \mathcal{J}_\nu[x(N)] \pm i\mathcal{Y}_\nu[x(N)]$ are the Hankel functions, $\mathcal{J}_\nu[x(N)]$ and $\mathcal{Y}_\nu[x(N)]$ are the first and second kind Bessel functions with $\nu = \sqrt{\alpha}$ and $x(N) = k_r e^{-N}$. Furthermore, the arbitrary constants A_1, A_2, B_1 and B_2 are constrained by the normalization condition (13) in the following manner

$$[(A_1 - A_2)(A_1 + A_2)] [B_1 \psi^{\frac{1}{2} + \nu} + B_2 \psi^{\frac{1}{2} - \nu}]^2 = \frac{\pi}{4}. \tag{20}$$

Choosing the generalized Bunch-Davies vacuum, given in this case by $A_1 = 0$ and $B_1 = 0$ in the above expression, we find that the only way to obtain a constant A_2 is setting $\nu = \sqrt{\alpha} = 1/2$. This gives $A_2 = i\sqrt{\pi}/(2B_2)$. Hence, the solution of (14) with the normalized condition (20), which ensures the quantization of φ , is given by

$$\xi_{k_r k_\psi}(N, \psi) = e^{-ik_\psi \psi} \tilde{\xi}_{k_r}(N), \tag{21}$$

being $\tilde{\xi}_{k_r}(N) i(\sqrt{\pi}/2) \mathcal{H}_{1/2}^{(2)} [k_r e^{-N}]$. This result possesses great importance and deserves a careful interpretation. The condition (20), which guarantees the normalization (and hence the quantization) of the field φ , requires that $\alpha = 2k_\psi^2 \psi^2$ to be a constant. This fact can be interpreted in the following manner: the field χ varies along the coordinate ψ by changing its momentum in that direction in such a way that the product $k_\psi \psi$ remains constant (this means that k_ψ decreases as ψ increases or viceversa). A consequence of this is that the coordinate ψ results to be irrelevant for the Fourier expansion of χ , but is relevant for $\varphi = (\psi_0/\psi)^2 e^{-3N/2} \chi(N, \vec{r})$.

Therefore, the field χ in equation (9) now simplifies to

$$\chi(N, \vec{r}, \psi) \chi(N, \vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int dk_\psi [a_{k_r k_\psi} e^{ik_r \vec{r}} \tilde{\xi}_{k_r}(N) + c.c.], \tag{22}$$

and thus the field φ is given by

$$\varphi(N, \vec{r}, \psi) = e^{-3N/2} \left(\frac{\psi_0}{\psi} \right)^2 \chi(N, \vec{r}), \quad (23)$$

with $\chi(N, \vec{r})$ given by equation (22). Note that exponentials $e^{\pm ik_\psi \psi}$ disappear in $\chi(N, \vec{r})$, which in turn implies that this field is independent of ψ . This is very important because $\varphi(N, \vec{r}, \psi)$, and hence gravity, propagates on the 3D spatially isotropic space $r(x, y, z)$, but not on the additional space-like coordinate ψ . This should explain why matter only propagates on the 4D submanifold described by the coordinates (N, \vec{r}) . Furthermore, expression (23) is a consequence of the normalization in (20), which gives that $\partial\varphi/\partial\psi = -2\psi^{-1}\varphi$, because α is a constant of the spectrum such that $k_\psi^2 = \psi^2/8$. This fact should be very important in the equation of motion for gravitons (described by some equation similar to (14)), because it implies a fixed mass and eliminates the exchange of extra light particles in the gravitational interactions. However, this is not the subject of this paper, but the reader can see a different mechanism to eliminate these light states in Ref. [3].

Throughout the following sections, we study the implications of this 5D dynamics as seen by 4D observers in a power-law expanding universe.

1.2 Effective 4D Power law expansion

In what follows we extend the formalism, recently introduced by M. Bellini [16], by studying the general case of a variable Hubble parameter. In this model, developed from a 5D vacuum, the expansion of the universe is governed by a scalar field, which has been sliding down its potential energy hill along the entire the evolution of the universe. The relevance of this model is that the effective 4D dynamics of the field φ only depends of the Hubble parameter h , which is a cosmological observable. Furthermore, the effective 4D potential $V(\varphi)$ has a geometrical origin. In particular, in this paper we study the inflationary expansion and the radiation dominated epoch. The later, rather than inflation, describes a decelerated expansion of the universe. We start by considering the metric (1) with the coordinates and frames transformations

$$t = \int \psi(N) dN, \quad R = r\psi, \quad L = \psi(N) e^{-\int dN/u(N)}, \quad (24)$$

$$U^N = \frac{u(N)}{\psi \sqrt{u^2(N) - 1}} \rightarrow \hat{U}^t = \frac{2u(t)}{\sqrt{u^2(t) - 1}}, \quad (25)$$

$$U^r = 0 \rightarrow \hat{U}^R = -\frac{2Rh}{\sqrt{u^2(t) - 1}}, \quad (26)$$

$$U^\psi = -\frac{1}{\sqrt{u^2(N) - 1}} \rightarrow \hat{U}^L 0, \quad (27)$$

such that, for the dynamical foliation $\psi(t) = 1/h(t)$ and $u(t) = 1/(1+q(t))$ (being $h(t)$ the effective Hubble parameter), we obtain an effective 4D Friedmann-Robertson-Walker metric

$$dS^2 \rightarrow ds^2 = \epsilon \left(dt^2 - e^{2 \int h(t) dt} dR^2 \right). \quad (28)$$

Here $q(t) = -\ddot{b}/\dot{b}^2$ is the deceleration parameter, the overdot denotes derivative with respect to the time and $b(t)$ is the scale factor of the universe [15]. The following geodesic dynamics is fulfilled in both reference systems¹

$$\frac{dU^C}{dS} = -\Gamma_{AB}^C U^A U^B, \quad g_{AB} U^A U^B = 1, \quad (29)$$

$$\frac{d\hat{U}^C}{dS} = -\hat{\Gamma}_{AB}^C \hat{U}^A \hat{U}^B, \quad \hat{g}_{AB} \hat{U}^A \hat{U}^B = 1. \quad (30)$$

The metric (28) has an effective 4D nonzero scalar curvature ${}^{(4)}\mathcal{R} = 6(\dot{h} + 2h^2)$ and a metric tensor with components $g_{\mu\nu}$. The absolute value of the determinant for this tensor is $|{}^{(4)}g| = (b/b_0)^6$. The Lagrangian density in this new frame was obtained in a previous work [17] and it has the form

$${}^{(4)}\mathcal{L} [\varphi(\vec{R}, t), \varphi_{,\mu}(\vec{R}, t)] = \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - V(\varphi), \quad (31)$$

where

$$V(\varphi) = -\frac{1}{2} g^{\psi\psi} \varphi_{,\psi} \varphi_{,\psi} \Big|_{\psi=h^{-1}} = \frac{1}{2} \left(\frac{\partial\varphi}{\partial\psi} \right)^2 \Big|_{\psi=h^{-1}} 2h^2(t) \varphi^2(t, \vec{R}, L) \quad (32)$$

is the effective 4D induced potential for the scalar field $\varphi(t, \vec{R}, L)$.

On the other hand, the 4D expectation values for energy density ρ and the pressure p are given by [14]

$$8\pi G \langle \rho \rangle = 3h^2, \quad (33)$$

$$8\pi G \langle p \rangle = -(3h^2 + 2\dot{h}). \quad (34)$$

where $G = M_p^{-2}$ is the gravitational constant and $M_p = 1.2 \times 10^{19} \text{ GeV}$ is the Planckian mass. Furthermore, the equation of state that describes the universe is $\langle p \rangle = -[1 - 2/(3u(t))] \langle \rho \rangle$, and from the hyperbolic condition $g_{AB} U^A U^B = 1$

¹For a more complete description see [17].

we obtain that the 4D evolution of the universe in terms of $b(t)$ is described by

$$1 + \frac{3}{\left(1 - \frac{\ddot{b}b}{b^2}\right)^2} = 4r^2 \left(\frac{b}{b_0}\right)^2. \tag{35}$$

This equation makes an analytical treatment a hard nut to crack. Thus, we restrict ourselves to the case where $u(t)$ is approximately a constant: $u(t) \simeq p$, where p describes the power of the expansion for the scale factor $b \sim t^p$.

During inflation, the universe is accelerated ($q < 0$) and $u(t) \gg 1$, thus $\epsilon = 1$. However, in epochs of decelerated expansion, $0 < u(t) < 1$ and $\epsilon = -1$ in the metric (1). The change of signature of ϵ means that dS^2 changes sign. This possibility was first considered by Davidson and Owen [18], who studied elementary particles as higher dimensional tachyons. In the model studied here, causality is affected during the inflationary epoch, but not when the expansion of the universe is decelerated. The possibility that causality is violated during inflation has been studied by many other authors as well [19]. In our case we have in mind an early accelerated universe (inflationary expansion), which at later times is decelerated. During inflation $dS^2 < 0$ so that causality is affected, but after inflation ceases (i.e., when the universe is decelerated), causality is restored, $dS^2 > 0$.

The expectation value for the energy density is

$$\langle \rho \rangle = \left\langle \frac{\dot{\varphi}^2}{2} + \frac{b_0^2}{2b^2} (\vec{\nabla}\varphi)^2 + V(\varphi) \right\rangle, \tag{36}$$

where the brackets denote the 4D expectation value.

Considering equation (6) and the transformation of coordinates (24), the effective 4D equation of motion for φ is

$$\ddot{\varphi} + \left(3h - \frac{\dot{h}}{h}\right) \dot{\varphi} - e^{-2 \int h(t) dt} \nabla_{R\varphi}^2 \varphi - \left[\frac{4}{\psi} \frac{\partial \varphi}{\partial \psi} + \frac{\partial^2 \varphi}{\partial \psi^2} \right] \Big|_{\psi=h^{-1}} = 0, \tag{37}$$

and the effective 4D value for $V'(\varphi)$ is

$$V'(\varphi) \Big|_{4D} = - \left[\frac{\dot{h}}{h} \frac{\partial}{\partial t} - 2h^2 \right] \varphi. \tag{38}$$

Now we apply the transformation

$$\varphi(t, \vec{R}) = \chi(t, \vec{R}) e^{-\frac{1}{2} \int \left(3h - \frac{\dot{h}}{h}\right) dt} = \chi(t, \vec{R}) e^{-\frac{3}{2} \int h dt} \left(\frac{h}{h_0}\right)^{1/2}, \tag{39}$$

in order for equation (37) to become a 4D Klein-Gordon equation for χ :

$$\ddot{\chi} - \left[e^{-2 \int_{t_0}^t h(t') dt'} \nabla_{R\chi}^2 + \frac{h^2}{4} + \frac{3}{4} \left(\frac{\dot{h}}{h}\right)^2 - \frac{1}{2} \left(\frac{\ddot{h}}{h}\right) \right] \chi = 0. \tag{40}$$

Here t_0 is the time at end of inflation. On the other hand we have

$$\chi(t, \vec{R}, L) \equiv \chi(t, \vec{R}) \frac{1}{(2\pi)^{3/2}} \int d^3 k_R \int dk_\psi \left[a_{k_R} e^{i\vec{k}_R \cdot \vec{R}} \tilde{\xi}_{k_R}(t) + c.c. \right] \delta(\vec{k}_\psi - \vec{k}_L), \quad (41)$$

where $L = \psi_0 = cte$. Hence the equation of motion for the time dependent modes $\tilde{\xi}_{k_R}(t)$ is

$$\ddot{\tilde{\xi}}_{k_R} + \left[k_R^2 e^{-2 \int h dt} - \frac{h^2}{4} - \frac{3}{4} \left(\frac{\dot{h}}{h} \right)^2 + \frac{1}{2} \left(\frac{\ddot{h}}{h} \right) \right] \tilde{\xi}_{k_R} = 0. \quad (42)$$

This means that the model excludes non-expanding cosmological models. Besides, all terms inside the brackets have a geometrical origin because they are induced by the fifth coordinate.

2 Some examples

To illustrate the formalism, we study some examples of power-law expanding universes which are relevant for cosmology.

2.1 Power-law inflation

We first consider a power law inflationary expansion for the universe

$$h(t) = \frac{p}{t}, \quad p > 1. \quad (43)$$

Equation (42) for the k_R -modes in power-law inflation is

$$\ddot{\tilde{\xi}}_{k_R} + \left[\frac{\alpha_0^2 t^{2(1-p)} - \beta^2}{t^2} \right] \tilde{\xi}_{k_R} = 0, \quad (44)$$

with $\alpha_0 = k_R t_0^p$, $\beta^2 = (p^2 - 1)/4$. The general solution of (44) is

$$\tilde{\xi}_{k_R}[x(t)] = \sqrt{t} \left(C_1 \mathcal{H}_\nu^{(1)}[x(t)] + C_2 \mathcal{H}_\nu^{(2)}[x(t)] \right), \quad (45)$$

being $\nu = p/[2(p-1)]$ and $x(t) = (\alpha_0 t^{1-p})/(p-1)$. Hence, when the normalization condition for $\tilde{\xi}_{k_R}(t)$ and the Bunch-Davies vacuum are both considered, i.e., $C_1 = 0$, $C_2 = i\sqrt{\pi/[4(p-1)]}$, we obtain

$$\tilde{\xi}_{k_R}(t) = i \sqrt{\frac{\pi}{4(p-1)}} \sqrt{t} \mathcal{H}_\nu^{(2)} \left[\frac{\alpha_0 t^{1-p}}{p-1} \right], \quad (46)$$

where the condition $\nu = 1/2$ is fulfilled only for $p \rightarrow \infty$ in inflationary models. Note that this result is very similar to the one obtained in the power law standard inflationary approach.

Once the time dependent 4D-modes $\tilde{\xi}_{k_R}(t)$ are known, we can calculate the effective 4D super Hubble squared fluctuations $\langle \varphi^2 \rangle$. For this purpose, we must remember the small argument limit for the second kind Hankel function: $\mathcal{H}_\nu^{(2)}[x] \simeq (x/2)^\nu / \Gamma(1 + \nu) - (i/\pi)\Gamma(\nu)(x/2)^{-\nu}$. Note that in this case $\nu < 0$ because $p > 1$, so we can use $\mathcal{H}_\nu^{(2)}[x] \simeq (x/2)^\nu / \Gamma(1 + \nu)$ and hence $\langle \varphi^2(\vec{R}, t) \rangle$ can be calculated on the infrared (IR) sector from the expression

$$\langle \varphi^2(t, \vec{R}) \rangle_{IR} = \frac{2^{\frac{p}{p-1}}(p-1)^{\frac{1}{p-1}}}{8\pi\Gamma^2\left[\frac{p-2}{2(p-1)}\right]} t_0^{2p} t^{-2p} \int_0^{\theta_{k_H}} \frac{dk_R}{k_R} k_R^{\frac{2p-3}{(p-1)}}, \quad (47)$$

where $\theta = k_{max}^{(IR)}/k_p \ll 1$ is a dimensionless constant (of order $\sim 10^{-4} - 10^{-3}$), $k_{max}^{(IR)} = k_H(t_*) = e^{\int h dt} h|_{t=t_*} p(t_*/t_0)^p t_*^{-1}$ is the wavenumber at the moment t_* , when the horizon enters, and k_p is the Planckian wavenumber (i.e., the scale we choose as a cut-off of all the spectrum). In other words, $k_H(t_*)$ is the wavenumber related to the Hubble radius in an expanding universe when the horizon enters.

2.2 Radiation dominated universe

A radiation dominated universe is described by a power $p = 1/2$, which is related with a Hubble parameter $h(t) = 1/(2t)$, such that equation (42) becomes

$$\ddot{\tilde{\xi}}_{k_R} + \left(\frac{? \alpha_1^2 t + \beta_1^2}{t^2} \right) \tilde{\xi}_{k_R} = 0, \quad (48)$$

with $\alpha_1 = k_R t_0^{1/2}$ and $\beta_1^2 = 3/16$. The general solution for this equation is

$$\tilde{\xi}_{k_R}(t) = G_1 \sqrt{t} \mathcal{H}_{\nu_1}^{(1)}[2\alpha_1 t^{1/2}] + G_2 \sqrt{t} \mathcal{H}_{\nu_1}^{(2)}[2\alpha_1 t^{1/2}], \quad (49)$$

being (G_1, G_2) constants and $\nu_1 = \sqrt{1 - 4\beta_1^2}$. Hence, when the normalization condition for $\xi_{k_R}(t)$ and the Bunch-Davies vacuum are considered, we obtain $G_1 = 0$ and $G_2 = \sqrt{\pi/2}$. Thus, the normalized solution for equation (48) is

$$\tilde{\xi}_{k_R}(t) = \sqrt{\frac{\pi}{2}} \sqrt{t} \mathcal{H}_{\nu_1}^{(2)}[2\alpha_1 t^{1/2}], \quad (50)$$

where in our case $\nu_1 = 1/2$. In this fashion, on super Hubble scales we obtain

$$\langle \varphi^2 \rangle_{IR} = \frac{\Gamma^2[1/2]}{4\pi^3} \left(\frac{t_0}{t} \right) \int_0^{\theta_{k_H}} \frac{dk_R}{k_R} k_R^2, \quad (51)$$

which, after integration yields

$$\langle \varphi^2 \rangle_{IR} = \frac{\theta^2 \Gamma^2[1/2]}{32\pi^3} t^{-2}. \quad (52)$$

Note the difference between the solutions of equations (44) and (48). Solutions of (44) are unstable outside the Hubble horizon and stable inside it. However, solutions of (48) are stable over the entire the spectrum. Thus, inflationary dynamics is necessary to explain why the universe is larger than the Hubble horizon.

3 Final Comments

In this work we have extended a previously introduced nonperturbative scalar field formalism for a noncompact KK theory to a power-law expanding universe. In particular, we have examined some relevant cases in cosmology; inflationary dynamics ($p > 1$) and radiation dominated (with $p = 1/2$) universes, respectively. We have found that the $\langle \varphi^2 \rangle_{IR}$ spectrum is scale invariant only during inflation for $p = 3/2$. This formalism has the advantage, with respect to the semi-classical approach studied in [17], that it provides an exact (nonperturbative) treatment for the dynamics of the inflaton field (the scalar field). Furthermore, it provides a consistent treatment for the effective 4D dynamics of the universe in models governed by a single scalar field. Note that the dynamics is induced geometrically through the potential in the frame $\hat{U}^t = 2u/(\sqrt{u^2 - 1})$, $\hat{U}^R = -2Rh/(\sqrt{u^2 - 1})$, $\hat{U}^L = 0$, where the fifth coordinate $L = \psi_0$ remains constant. This constant is given by the Hubble horizon at the end of inflation. The potential is quadratic in φ but its squared mass depends on the rate of expansion of the universe (the Hubble parameter). In epochs where the expansion is accelerated the signature of the squared mass is positive, but when the expansion is decelerated it shifts to a negative signature.

Since tensor metric fluctuations h_{AB} are not the subject of this paper, in our analysis we excluded possible short-range modifications of Newton's gravitational law due to the 5D graviton propagator. For a discussion about astrophysical implications of the existence of extra dimensions see [20, 21].

Finally, the theory can explain why inflationary dynamics is necessary to explain the existence of an universe larger than the Hubble horizon, because the modes related to cosmological scales of the quantum field χ grows superluminally for a scale factor $b \sim t^p$, with $p > 1$. However, for a radiation dominated universe the modes are stable over the entire the spectrum. This is a remarkable result of this paper: *during the radiation dominated epoch the squared φ -fluctuations are stable over all scales, rather in models where the universe is (as inflation) accelerated.* It is evident the difference between the

result (52) and those obtained in [22], where $\langle \varphi^2 \rangle_{IR} \sim t^{-3/2}$. However, as was demonstrated in [22] the stochastic approach is only valid for $p \gg 2$.

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References

- [1] N. Akdani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436**, 257 (1998).
- [2] D. Youm, Phys. Rev. **D62**, 084002 (2000); R. Maartens, Phys. Rev. **D62**, 084023 (2000); A. Chamblin, Class. Quantum Grav. **18**, L-17 (2001); P. S. Wesson, B. Mashhoon, H. Liu, W. N. Sajko, Phys. Lett. **B456**, 34 (2001); J. Ponce de Leon, Phys. Lett. **B523**, 311 (2001).
- [3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
- [4] T. Kaluza, Sitz. Preuss. Akad. Wiss. Phys. Math. **K1**, 966 (1921); O. Klein, Zeits. Phys. **37**, 895 (1926).
- [5] J. M. Overduin and P. S. Wesson, Phys. Rep. **283**, 303 (1997).
- [6] P. S. Wesson, *Space, Time, Matter* (World Scientific, Singapore, 1999).
- [7] S. S. Seahra and P. S. Wesson, Class. Quant. Grav. **19**, 1139 (2002).
- [8] L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
- [9] J. E. Campbell, *A course of differential geometry*, Oxford, Clarendon Press.
- [10] L. Magaard, *Zur einbettung riemannscher raume in Einstein-raume und konforeuclidische raume*, PhD thesis, (Kiel); E. Anderson, F. Dahia, J. Lidsey and C. Romero, J. Math. Phys. **44**, 5108 (2002); J. Lidsey, C. Romero, R. Tavakol and S. Rippl, Class. Quant. Grav. **14**, 865 (1997); C. Romero, R. Tavakol and R. Zalaletdinov, Gen. Rel. Grav. **28**, 265 (1996).
- [11] J. Dreitlein, Phys. Rev. Lett. **33**, 1243 (1974).

- [12] A. Guth, *Phys. Rev.* **D23**, 347 (1981).
- [13] C. Wetterich, *Nucl. Phys.* **B302**, 668 (1988); P. J. E. Peebles and B. Ratra, *Ap. J. Lett.* **325**, L17 (1988); R. R. Caldwell, R. Dave and P. J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998); C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, *Phys. Rev. Lett.* **85**, 4438 (2000).
- [14] D. S. Ledesma and M. Bellini, *Phys. Lett.* **B581**, 1 (2004).
- [15] J. E. Madriz Aguilar and M. Bellini, *Eur. Phys. J.* **C38**, 367 (2004).
- [16] M. Bellini, *Phys. Lett.* **B609**, 187 (2005).
- [17] E. Madriz Aguilar and M. Bellini, *Phys. Lett.* **B596**, 116 (2004).
- [18] A. Davidson, D. A. Owen, *Phys. Lett.* **B177**, 77 (1986).
- [19] see for example M. Sami, P. Chingangbam, T. Qureshi, *Phys. Rev.* **D66**, 043530 (2002).
- [20] H. Liu, J. M. Overduin, *Astrophys. J.* **538**, 386 (2000).
- [21] S. S. Seahra, *Phys. Rev.* **D65**, 124004 (2002).
- [22] M. Anabitarte, J. E. Madriz Aguilar, M. Bellini. *Eur. Phys. J.* **C45**, 249 (2006).

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