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New Approach to Mixture of the Adomian Decomposition and Homotopy Perturbation Method for Approximate and Analytical Solution of Integral Equations and Fractional Differential Equations

Esmail Hesameddini

Faculty of Science Shiraz University of Technology, Shiraz, Iran
P.O.Box 31371555
Hesameddini@sutech.ac.ir

Habibolla Latifizadeh

Faculty of Science Shiraz University of Technology, Shiraz, Iran
H.latifi.62.Math@ sutech.ac.ir

Abstract

Nonlinear phenomena play a crucial role in applied mathematics and physics. Analytic solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining Analytic solution of nonlinear evolution equations have been proposed. In this paper a combination of the Homotopy Perturbation method (HPM) and Adomian decomposition method is employed for solving nonlinear integral equation and comparison is made between the other methods. The results reveal that the proposed method is very effective and simple and leads to accurate, approximately convergent solutions to linear and nonlinear equations.

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Keywords: Homotopy perturbation method; Adomian Decomposition method; integral equations; Fractional differential equations

1. Introduction

There has recently been much attention devoted to the search for reliable and more efficient solution methods for nonlinear equations physical phenomena in various fields of science and engineering, for this reason, this paper outline is a reliable mixture between two powerful methods. At the first, the Homotopy perturbation Method (HPM), one of the methods which have received much attention and, proposed by He [8, 11, 12] this does not require a small parameter in the equation contrast to the traditional perturbation methods. In He's Homotopy method with an embedding parameter $p \in [0,1]$ is constructed. The Homotopy perturbation, get to the solution, with much less computational work, [8-10], unlike the traditional numerical methods, this method is in principle based on Taylor series with respect to an embedding parameter so, mathematically speaking, Homotopy perturbation method itself is also a kind of generalized Taylor technique, besides, this method can give very well approximations by means of a few terms, if initial guess and auxiliary linear operator are good enough.

The second is Adomian decomposition method which, in the beginning of the 1980s, a so-called Adomian decomposition method was introduced by Adomian [1-3] for solving the nonlinear problems. It is well known that this method avoids linearization and provides an efficient numerical solution with high accuracy [4, 7, 14].

In this paper, the proposed method (MHA) has been successfully applied to finding the solutions of generalized nonlinear equation; and the obtained solutions are compared with those of ADM and HPM. The results show that MHA is a powerful mathematical tool for solving linear and nonlinear differential equations, and therefore, can be widely applied in engineering problems.

2. Basic concepts of the Homotopy-perturbation method

To explain this method, let us consider the following function;

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

With the boundary conditions of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma. \tag{2}$$

Where ***A, B, f (r),*** are general differential operator, a boundary operator, known analytical function and the boundary of the domain respectively. Generally speaking, the operator ***A*** can be divided into a linear part ***L*** and a nonlinear part ***N***. Esq. (1) can therefore is written as;

$$L(u) + N(u) - f(r) = 0, \tag{3}$$

By the Homotopy technique, we construct a Homotopy $v(r,p): \Omega \times [0,1] \rightarrow R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

Where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eqs (4) and (5) we will have;

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{7}$$

The changing process of " p "from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$.In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called Homotopy. According to the HPM, we can first use the embedding parameter " p "as a small parameter, and assume that the solutions of Esq. (4) and (5) can be written as a power series" p "in;

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{8}$$

Setting" $p = 1$ "a result in the approximate solution of Esq. (3) is;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{9}$$

The combination of the perturbation method and the Homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (9) is convergent for most cases. However, the rate convergent depends on non-linear term $A(v)$.

3. The Mixture of Adomian decomposition method and Homotopy perturbation Method

In this paper, using the following integral equations such that, is given by;

$$u(x) = \lambda\int_a^x k(x,t)\,N(u(t))dt + g(x), \qquad c \le x \le d \tag{10}$$

Applying by the same time the Homotopy perturbation method and the Adomian decomposition method for integral equations of Eq. (10) therefore, at first we construct the Homotopy perturbation relation as the following form;

$$H(u,p) = p(L(u)) + (1-p)F(u) = 0, \tag{11}$$

So that

$$L(u) = u(x) - \lambda\int_a^x k(x,t)\,N(u(t))dt - g(x), \tag{12}$$

$$F(u) = u(x) - g(x).$$

While $g(x)$ is supposed to be;

$$g(x) = g_0(x) + pg_1(x) + p^2 g_2(x) + \cdots \tag{13}$$

Now, suppose that $N(u(x))$ has one of the following forms, we use the Taylor expansion of the non-linear term;

$$N(u(x)) = e^{u(x)} = 1 + u(x) + \frac{u^2(x)}{2!} + \frac{u^3(x)}{3!} + \cdots \tag{14}$$

$$N(u(x)) = \sin(u(x)) = u(x) - \frac{u^3(x)}{3!} + \frac{u^5(x)}{5!} \pm \cdots$$

However, in practice, all terms of the above series cannot be used, so using of the truncated series, i.e. $N(u) = \sum_{h=0}^{n} N_h(u)$ so that, N_h 's are linear and nonlinear terms with respect to $u(x)$, already by applying the Adomian's polynomial for $N_h(u(x))$ to obtaining the coefficients of "p^k" in means of Homotopy perturbation method, which is;

$$A_h^k(u_0,\cdots,u_k) = \frac{1}{k!}\Big[\frac{d^k}{d\lambda^k} N_h\Big(\sum_{j=0}^{\infty}\lambda^j u_j\Big)\Big]_{\lambda=0} \tag{15}$$

Where

$$N_h = \sum_{i=0}^{\infty} A_h^i(u_0,\cdots,u_k) \tag{16}$$

So that, the terms A_h^k, the coefficients of "p^k" is the Adomian polynomial for $N_h(u)$ .It is noteworthy that, If $N_h(u) = u^n$ (Polynomial of degree "n"), then coefficients of "k^{th}" order of "p" has the following form;

$$A_h^k(u_0,\cdots,u_k)=\sum_{r_1=0}^{k}u_{r_1}(x)\sum_{r_2=0}^{k-r_1}u_{r_2}(x)\sum_{r_3=0}^{k-r_1-r_2}u_{r_3}(x)\cdots \tag{17}$$

$$\cdots\sum_{r_{n-1}=0}^{k-\sum_1^{n-2}r_t}u_{r_{n-1}}(x)\;u_{\left(k-\sum_{j=1}^{n-1}r_j\right)}(x).$$

For the especial case $n=3:\mathrm{N}_h(\mathrm{u}(x))=u^n(x)$, and one can obtain $A_h^k, k=0,...,3$ as follows; (18)

$$A_h^0 : u_0^3(x),$$

$$A_h^1 : 3u_0^2(x)u_1(x),$$

$$A_h^2 : 3u_0^2(x)u_2(x)+3u_1^2(x)u_0(x),$$

$$A_h^3 : 3u_0^2(x)u_3(x)+6u_0(x)u_1(x)u_2(x)+u_1^3(x),$$

$\vdots$

Therefore by substituting Esq. (13), (14), into Esq. (12), and using (16) for each nonlinear term, $u_i(x)$ 'swill be obtained as follows; (19)

$$u_k(x)=g_k(x)+\lambda\int_a^x k(x,t)\{\textstyle\sum_{i=0}^n A_i^{k-1}(u_0,\cdots,u_{k-1})(t)\}dt.$$

Therefore, in view of the Homotopy Perturbation method one can have;

$$u=\lim_{p\to 1}\sum_{i=0}^{\infty}p^i u_i = u_0+u_1+u_2+\cdots \tag{20}$$

Where by appropriate choice of $g_i(x)$ one can have the best approximation of the desirable solution of the problems. To give a clear overview of our study and to illustrate the reliability and performance of the above discussed technique, the following examples are considered.

4. Illustration of some examples

Example (4.1): Consider the following problem;

$$y'(x)+y^2(x)=1,\quad x>0 \tag{21}$$

With initial condition $y(0)=0$.

Converting this nonlinear differential equation to the integral equation as follows;

$$y(x) = x - \int_0^x y^2(t)dt. \qquad (22)$$

So that, the Esq. (21), (22) are equivalent. By using the proposed method (MHA), we construct the Homotopy perturbation relation as the following form;

$$H(y,p) = y(x) - x + p\int_0^x \overbrace{y^2(t)}^{N(y)}dt = 0. \qquad (23)$$

Therefore, one can obtain $y_k(x)$'s as;

$$y_k(x) = g_k(x) + \int_0^x \left(A_{k-1}(y_0,\cdots,y_{k-1})(t)\right)dt. \qquad (24)$$

The first few terms of the solution of $y(x)$ is given by regularly computed coefficients powers of "p"as follows;

$$p^0 : y_0(x) = x, \qquad (25)$$

$$p^1 : y_1(x) = -\int_0^x y_0^2(t)dt = -\frac{x^3}{3},$$

$$p^2 : y_2(x) = -\int_0^x (2y_0(t)y_1(t))dt = \frac{2}{15}x^5,$$

$$p^3 : y_3(x) = -\int_0^x (2y_0(t)y_2(t) + y_1^2(t))dt = -\frac{1}{315}x^7,$$

$$\vdots$$

Therefore the Taylor expansion of the exact solution of the problem that is $y(x) = \tanh(x) = (e^{2x} - 1)/(e^{2x} + 1)$, will be established.

Example (4.2): considering the following equation;

$$u(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xu^3(t)dt. \qquad (26)$$

Doing the same procedure, one can obtain the following Homotopy perturbation notation as;

$$H(u,p) = u(x) - g(x) - p\int_0^x \overbrace{xu^3(t)}^{N(u)}dt = 0, \qquad (27)$$

So that;

$$\begin{aligned} \mathbf{g(x)} &= \mathbf{g_0(x) + pg_1(x) + p^2g_2(x) + p^3g_3(x)} + \cdots = \\ &= \mathbf{e^x + p\left(\frac{1}{3}x - \frac{1}{3}xe^{3x}\right) + 0p^2 + 0p^3} + \cdots \end{aligned} \qquad (28)$$

Therefore, $u_0(x)$'s will be suggested as;

$$u_k(x) = g_k(x) + \lambda \int_0^x x\{A_{k-1}(u_0, \cdots, u_{k-1})\}dt. \qquad (29)$$

The first few terms of the solution $u(x)$ is given by regularly computed coefficients powers of "p"in the following form;

$$p^0 : u_0(x) = e^x, \qquad (30)$$

$$p^1 : u_1(x) = \frac{1}{3}x - \frac{1}{3}xe^{3x} + \int_0^x xu_0^3(t)dt = 0,$$

$$p^2 : u_2(x) = \int_0^x x\left(3u_0^2(t)u_1(t)\right)dt = 0,$$

$$\vdots$$

Using Eqs. (8), (9) the exact solution is;

$$y(x) = \lim_{p\to 1}\left(p^0 u_0(x) + p^1 u_1(x) + p^2 u_2(x) + \cdots\right) = e^x. \qquad (31)$$

In this case, by using the best choice of $g_i(x)$, one can obtain the exact solution after two iterations.

Exemple (4.3) : Consider the following third order ordinary differential equation;

$$y'''(x) + y'(x) + y^2(x) = x^2 + 1, \qquad (32)$$

Subject to the initial conditions;

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$$

Eq. (32) can be transformed into a system of the first-order ordinary differential equations by introducing;

$$y(x) = y_1(x), \quad y_1'(x) = y_2(x), \quad y_2'(x) = y_3(x). \qquad (33)$$

Then writing the n^{th} order initial value problems Eq. (32) as a system of ordinary differential equations;

$$y_1'(x) = y_2(x), \qquad (34)$$

$$y_2'(x) = y_3(x),$$

$$y_3'(x) = x^2 + 1 - (y_2(x) - y_1^2(x)),$$

With the initial conditions;

$$y_1(x) = 0,\ y_2(x) = 0,\ \ y_3(x) = 1.$$

And then by proposed Homotopy perturbation method one obtain;

$$(35)$$

$$H_1(y_1, y_2, y_3, p) = y_1(x) - g_1(x) - p\int_0^x y_2(t)dt = 0,$$

$$H_2(y_1, y_2, y_3, p) = y_2(x) - g_2(x) - p\int_0^x y_3(t)dt = 0,$$

$$H_3(y_1, y_2, y_3, p) = y_3(x) - g_3(x) - p\int_0^x \left(y_2(t) - y_1^2(t)\right)dt = 0,$$

Where $g_i(x), i = 1,...,3$ have the following forms;

$$g_1(x) = 0 + 0p + 0p^2 + 0p^3 + \cdots, \tag{36}$$

$$g_2(x) = 1 + 0p + 0p^2 + 0p^3 + \cdots,$$

$$g_3(x) = x + p\frac{x^3}{3} + 0p^2 + 0p^3 + \cdots$$

Using the same way of the proposed method, the solution will be obtained by relations (8), (9) and (35) as;

$$\mathbf{y}(x) = \mathbf{lim}_{p\to 1}\left(\mathbf{y}_{10}(\mathbf{x}) + \mathbf{p}\mathbf{y}_{11}(\mathbf{x}) + \mathbf{p}^2\mathbf{y}_{12}(\mathbf{x}) + \cdots\right) = \mathbf{x}. \tag{37}$$

Also it worth be nothing that, there is a noise term at each terms of $y_{1i}(x)$'s.

Example (4.4): Consider the following nonlinear differential equation;

$$y'(x) - y(x) + y^2(x) = 0, \tag{38}$$

Subject to the initial conditions,

$$y(0) = 0.5.$$

By using the proposed method and constructing the Homotopy perturbation, we have;

$$H(y,p) = y(x) - g(x) - p\int_0^x \overbrace{\left(y(t) - y^2(t)\right)}^{N(y)}dt \tag{39}$$

So that;

$$\begin{cases} g(x) = g_0(x) + pg_1(x) + p^2g_2(x) + \cdots = \dfrac{1}{2} + 0p + 0p^2 + \cdots \\ N(y(x)) = y(x) - y^2(x) = N_0(y) + N_1(y). \end{cases} \tag{40}$$

Therefore, $y_k(x)$'s will be obtained as;

$$y_k(x) = g_k(x) + \lambda\int_a^x \{y_{k-1}(t) + A_{k-1}(y_0, \cdots, y_{k-1})\}\,dt. \tag{41}$$

Some coefficients of powers "p"are as the following;

$$p^0 : y_0(x) = \frac{1}{2}, \tag{42}$$

$$p^1 : y_1(x) = \int_0^x (y_0(t) - y_0^2(t))dt = \frac{x}{4},$$

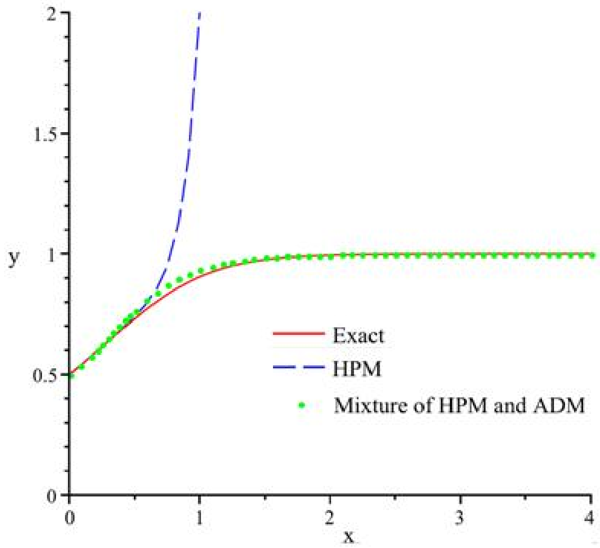
$$p^2 : y_2(x) = \int_0^x (y_1(t) - 2y_0(t)y_1(t))dt = 0,$$

$$p^3 : y_3(x) = \int_0^x (y_2(t) - \{2y_0(t)y_1(t) + y_1^2(t)\})dt = \frac{-x^3}{48},$$

$$p^{2m} : y_{2m}(x) = 0, \qquad m \geq 2.$$

⋮

In the following figure, which is calculated for five iterations of Eq. (42), one can see the efficiency of the method for finding the approximate solution in compare with HPM method.



Example (4.5): Consider the following general nonlinear equation as;

$$u(x) = \lambda \int_a^x k(x, t)\, e^{u(t)}\, dt + g(x), \qquad c \leq x \leq d \tag{43}$$

Since, the integration of the nonlinear term in Eq. (43) is not easily evaluated, thus one way is that replacing the nonlinear term with a series of finite components as;

$$N(u) = e^{u(t)} = \overbrace{1}^{N_0} + \overbrace{u(t)}^{N_1} + \overbrace{\frac{(u(t))^2}{2!}}^{N_2} + \cdots + \overbrace{\frac{(u(t))^n}{n!}}^{N_n} = \sum_{s=0}^{n} N_s(u). \tag{44}$$

Under this assumption, therefore, we consider the following scheme;

$$L(u) = u(x) - \lambda \int_a^x \left\{\sum_{s=0}^{n} N_s(u)\right\} k(x,t)dt - g(x). \tag{45}$$

Then by applying the previous manner and using the proposed method (MHA), one can obtain the solution such that;

$$g(x) = g_0(x) + pg_1(x) + p^2 g_2(x) + \cdots$$

And by using the following relation;

$$u_k(x) = g_k(x) + \lambda \int_a^x [N_0(u) + N_1(u) + A_2^{k-1} + \cdots + A_n^{k-1}] k(x,t)\,dt, \tag{46}$$

The solution of Eq. (43) will be established as;

$$y(x) = \lim_{p\to 1}(y_0(x) + py_1(x) + p^2 y_2(x) + \cdots). \tag{47}$$

5. Definitions

At first, some basic definitions and properties of fractional calculus theory, which is used in this paper, will be presented.

Definition.5.1. A real function $f(x), x > 0,$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number; $p(> \alpha)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space;

$$f \in C_\mu^m \text{ if and only if } f^{(m)} \in C_\mu, \quad m \in N.$$

Definition.5.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0,$ of a function, $f \in C_\mu, \mu \geq -1,$ is defined as;

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{48}$$

$$I^0 f(x) = f(x).$$

According to [17,18];

$$I^\alpha I^\beta f(x) = I^{(\alpha+\beta)} f(x), \tag{49}$$

$$I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x),$$

$$I^\alpha x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} x^{\lambda+\alpha}.$$

Where $\alpha, \beta \geq 0, x > 0$ and $\lambda > -1.$

Definition.5.3: The fractional derivative of the Caputo sense is defined as;

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{m}(t)\,dt, \tag{50}$$

$$m-1 \le \alpha \le m, \quad m \in \mathbf{N}, \qquad x > 0, \quad f \in C_{-1}^{m}.$$

$$D^{\beta} x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)} x^{\lambda-\beta},$$

$$I^{\beta} D^{\beta} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^{k}}{k!}.$$

In the following example, the initial value problem of the fractional differential equation of order μ, such that $m-1 < \mu \le m$, will be considered.
Also the analytical results on existence and uniqueness of the solutions to fractional differential equations have been investigated by many authors [6, 16, 17].

Example (5.1): Let us first consider the following nonlinear differential equation of the fractional order;

$$D^{\mu} y(x) = y^{2}(x) + 1, \tag{51}$$

$$m-1 < \mu \le m, \quad 0 < x < 1,$$

Subject to the initial conditions;

$$y^{(k)}(0) = 0, \qquad k = 0, \ldots, m-1.$$

First, by applying the operator I^{μ}, the inverse operator D^{μ} to both sides of Eq. (51), and using initial conditions, the following Homotopy perturbation will be obtained;

$$H(y,p) = y(x) - I^{\mu}(1) - pI^{\mu}\left(y^{2}(x)\right) = 0. \tag{52}$$

Computing the coefficients of power of p as before, therefore, one can have

$$\begin{aligned} p^{0} &: y_{0}(x) = I^{\mu}(1) = C_{0} x^{\mu}, \\ p^{1} &: y_{1}(x) = I^{\mu}\left(y_{0}^{2}(x)\right) = C_{1} x^{3\mu}, \\ p^{2} &: y_{2}(x) = I^{\mu}\left(2 y_{0}(x) y_{1}(x)\right) = C_{2} x^{5\mu}, \\ p^{3} &: y_{3}(x) = I^{\mu}\left(2 y_{0}(x) y_{2}(x) + y_{1}^{2}(x)\right) = C_{3} x^{7\mu}, \\ p^{4} &: y_{4}(x) = I^{\mu}\left(2 y_{0}(x) y_{3}(x) + 2 y_{1}(x) y_{2}(x)\right) = C_{4} x^{9\mu}, \\ &\vdots \end{aligned} \tag{53}$$

And these constant are identified as follows;

$$C_0 = \frac{1}{\Gamma(\mu+1)},$$

$$C_1 = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} C_0^2,$$

$$C_2 = \frac{\Gamma(4\mu+1)}{\Gamma(5\mu+1)} \left(2C_0C_1\right),$$

$$C_3 = \frac{\Gamma(6\mu+1)}{\Gamma(7\mu+1)} \left(2C_0C_2 + C_1^2\right),$$

$$C_4 = \frac{\Gamma(8\mu+1)}{\Gamma(9\mu+1)} \left(2C_0C_3 + 2C_1C_2\right),$$

⋮

The values are computed only for the case $\mu = 1$. Therefore, by using the proposed method (MHA) the exact solution of the equation in the closed form will be suggested as $y(x) = \tan(x)$.

Conclusion

In this paper, a mixture of the Adomian decomposition and Homotopy Perturbation Method (MHA) is introduced to solve nonlinear equations, integral equations, differential equation of fractional order that is leading to a wide application in science and engineering. We do the computation of more complex nonlinear terms with the help of MAPLE software.

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